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hand in on 12th June 2015 before lecture

Advanced Statistical Physics II – Problem Sheet 8

Problem 1 – **Liouville equation**

a) Prove

$$\int dpdq\rho(\tilde{q},\tilde{p},t+\tau;q,p,t) = \rho(\tilde{q},\tilde{p},t+\tau), \tag{1}$$

where $\rho(\tilde{q},\tilde{p},t+\tau;q,p,t)$ is the joint probability distribution of finding system at $\tilde{q},\tilde{p}$ at time $t+\tau$, and at q,p at time t. For this use Bayes' theorem and the Liouville propagators $e^{-\tau\mathcal{L}}$ and $e^{-t\mathcal{L}}$.

b) Show that if $\rho_0(q,p)$ is a stationary distribution then

$$e^{-t\mathcal{L}(q,p)}\rho_0(q,p) = \rho_0(q,p), \tag{2}$$

where $\mathcal{L}(q,p)$ is the Liouville operator.

c) Consider a particle trapped in a harmonic potential $V(q) = kq^2/2$, in one dimension. The initial probability distribution is $\rho(q,p,t=0) = \delta(q-q_0)\delta(p-p_0)$.

   i) Sketch a phase space trajectory using suitable start parameters.

   ii) By using the Liouville equation calculate $\langle p\rangle$, $\langle p^2\rangle$, $\langle q\rangle$ and $\langle q^2\rangle$.

Problem 2 – **Functional Taylor expansion**

The functional Taylor expansion for a functional $G[f]$ reads

$$G[f] = G[0] + \int dx \left.\frac{\delta G[f]}{\delta f(x)}\right|_{f=0} f(x) + \frac{1}{2!}\int dxdx' \left.\frac{\delta^2 G[f]}{\delta f(x)\delta f(x')}\right|_{f=0} f(x)f(x') + \dots \tag{3}$$

Expand the functional $G[f] = \int dx\,\left[f(x) + f(x)^2\right]$ around $f = 0$, and show that the expansion reproduces the original functional $G[f]$.

Problem 3 – **Response function**

Consider a harmonic oscillator subject to a time-dependent driving force $F(t)$. The equation of motion of the oscillator position $x(t)$ is given as

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = F(t), \tag{4}$$

where γ is the friction constant, ω_0 is the natural frequency of the osillator.

a) Show that the force-position response function is $\chi(\omega) = (-\omega^2 + i\gamma\omega + \omega_0^2)^{-1}$. For this use the Fourier transiform $\chi(t) = \int d\omega e^{i\omega t}\chi(\omega)/2\pi$. Find the force-position relation in the zero-frequency limit. Discuss overdamped and underdamped cases.

b) Sketch plots of the real part and the imaginary part of the response function $\chi(\omega)$ with $\omega_0 = 1$ and $\gamma = 0.2$. Sketch plots of the real part and the imaginary part of the response function $\chi(\omega)$ with $\omega_0 = 1$ and $\gamma = 4$.

c) Consider a force $F(t) = Re[F_0 e^{-i\Omega t}]$. Calculate the absorbed energy averaged over one period,

$$\mathcal{W} = \frac{\Omega}{2\pi} \int_0^{2\pi/\Omega} dt F(t) \frac{dx}{dt}, \tag{5}$$

and relate $\mathcal{W}$ to the imaginary part of the response function $\chi(\Omega)$. For this use the Fourier transform of $F(t)$. At what frequency Ω is the absorbed energy maximized?