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Hand in on January 16th 2017 before the lecture

Advanced Statistical Physics II – Problem Sheet 10

Problem 1 – **Mean squared displacement**

The Langevin equation

$$m\frac{\mathrm{d}}{\mathrm{d}t}v(t) = -\gamma v(t) + F_R(t)\,, \tag{1}$$

where m denotes the particle mass, $v(t)$ the velocity, γ the friction coefficient and $F_R(t)$ the random force, is an inhomogenous linear differential equation of first order. On the seventh problem sheet and in the lecture we have shown that the solution is

$$v(t) = v(0)\mathrm{e}^{-\gamma t/m} + \frac{1}{m}\int_0^t \mathrm{d}t'\mathrm{e}^{-\gamma(t-t')/m}F_R(t')\,. \tag{2}$$

The random force F_R is assumed to be Gaussian white noise, which has the properties

$$\langle F_R(t)\rangle = 0 \qquad \text{and} \qquad \langle F_R(t_1)F_R(t_2)\rangle = 2\gamma k_B T\delta(t_1 - t_2)\,, \tag{3}$$

where $\langle \dots \rangle$ denotes the expectation value over the random force.

a) (5P) Using this solution of the differential equation, show that the mean squared displacement is

$$\left\langle \Delta x(t)^2\right\rangle = \left\langle (x(t) - x(0))^2\right\rangle = \frac{2k_B T}{\gamma}\left(t - \frac{m}{\gamma} + \frac{m}{\gamma}\mathrm{e}^{-\gamma t/m}\right)\,, \tag{4}$$

where $x(t)$ is the position of the particle at a time $t > 0$ and T is the temperature.
Hint: First, express the mean squared displacement in terms of the velocity $v(t)$ instead of the position $x(t)$. Solve the resulting integrals making use of Eq. 3 and the equipartition theorem.

b) (2P) Discuss the limits $t \gg m/\gamma$ and $t \ll m/\gamma$.

c) (2P) Starting from the Einstein equation $\left\langle \Delta x(t)^2\right\rangle = 2Dt$, where D is the diffusion constant, show that

$$D = \int_0^t \mathrm{d}t'\,\langle v(t')v(0)\rangle \tag{5}$$

Problem 2 – **Langevin equation of harmonic oscillator**

(4P) Consider the Langevin equation from problem 1 with an additional force stemming from a harmonic potental:

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) = -\gamma\frac{\mathrm{d}}{\mathrm{d}t}x(t) - Kx(t) + F_R(t)\,, \tag{6}$$

with the spring constant K of the harmonic potential. In the lecture it has been shown that the expectation value of $x(t)^2$ can be written as

$$\left\langle \Delta x(t)^2\right\rangle = \frac{k_B T}{\pi}\int_0^t \mathrm{d}\omega\,\frac{\tilde{\chi}''(\omega)}{\omega} = \frac{\gamma k_B T}{\pi}\int_0^t \mathrm{d}\omega\,\frac{1}{(K - \omega^2 m)^2 + \gamma^2\omega^2} \tag{7}$$

which has been solved using the Kramers-Kronig relations. Here, compute $\langle \Delta x(t)^2 \rangle$ by explicit residual calculus.

Problem 3 – **Diffusion equation**

Consider the probability distribution of a particle in one dimension governed by the diffusion equation

$$\frac{\partial c(x,t)}{\partial t} = D\frac{\partial^2}{\partial x^2}c(x,t). \tag{8}$$

a) (1P) Express $c(x,t)$ in equation (8) through its Fourier transform $\tilde{c}(q,t)$ to show that equation (8) implies

$$\frac{\partial}{\partial t}\tilde{c}(q,t) = -Dk^2\tilde{c}(q,t). \tag{9}$$

b) (2P) Find the stationary distribution $\tilde{c}(q, t \to \infty) = \tilde{c}_{\text{stat.}}(q)$ and from that derive the distribution in real space $c_{\text{stat}}(x)$.

c) (2P) Solve the diffusion equation (8) for a particle initially localized at x_0, i.e. with initial condition

$$c(x,0) = c_0(x) \equiv \delta(x - x_0). \tag{10}$$

d) (2P) Calculate the average position $\langle x(t)\rangle$ and the mean-squared displacement $\langle (x(t) - x(0))^2\rangle$ of the diffusing particle.