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Hand in on January 23th 2017 before the lecture

Advanced Statistical Physics II – Problem Sheet 11

Problem 1 – **Characteristic function**

Given a probability density function $\rho(\vec{x})$, the Fourier transform

$$\rho(\vec{k}) = \langle e^{i\vec{k}\cdot\vec{x}} \rangle = \int e^{i\vec{k}\cdot\vec{x}} \rho(\vec{x})\, \mathrm{d}^N x \tag{1}$$

is called **characteristic function** or **moment generating function**.

a) (3P) Show that

$$\langle x_{i_1} ... x_{i_m} \rangle = \left[\frac{\partial}{\partial(ik_{i_1})} \cdots \frac{\partial}{\partial(ik_{i_m})} \rho(\vec{k}) \right] \Bigg|_{\vec{k}=0} , \tag{2}$$

i.e. all the moments of the distribution can be calculated as derivatives of the characteristic function.

Remark: This is why it is also called moment generating function.

b) (2P) Consider the N-dimensional Gaussian distribution

$$\rho(\vec{x}) = \frac{1}{\sqrt{(2\pi)^N \det(\sigma)}} \exp\left[-\frac{1}{2} (\vec{x} - \vec{\alpha})^T \sigma^{-1} (\vec{x} - \vec{\alpha}) \right] , \tag{3}$$

where $\vec{\alpha} \in \mathbb{R}^N$ and σ is a symmetric $N \times N$ matrix whose eigenvalues are all positive. Calculate $\langle \vec{x} \rangle$, $\langle x_i x_j \rangle$ using formula (2) and use these results to calculate $\langle (x_i - \langle x_i \rangle)(x_j - \langle x_j \rangle) \rangle$.

Hint: Remember problem 3 on sheet 3.

Problem 2 – **Green's function for the Fokker-Planck equation**

Consider the Fokker-Planck equation

$$\frac{\partial}{\partial t} \rho(\vec{x}, t) = \mathcal{D} \rho(\vec{x}, t) , \tag{4}$$

where the operator $\mathcal{D}$ is defined by its action on an arbitrary function $f(\vec{x}, t)$ as

$$\mathcal{D} f = -\left(\frac{\partial}{\partial \vec{x}} \right)^T \Theta\, \vec{x} f + \left(\frac{\partial}{\partial \vec{x}} \right)^T B \frac{\partial}{\partial \vec{x}} f \equiv -\frac{\partial}{\partial x^i} \Theta_{ij} x_j f(\vec{x}, t) + B_{ij} \frac{\partial^2}{\partial x^i\, \partial x^j} f(\vec{x}, t) \tag{5}$$

with constant $n \times n$ matrices Θ, B.

We want to solve this equation for an arbitrary initial distribution $\rho_0(\vec{x}) \equiv \rho(\vec{x}, 0)$.

a) (2P) Show that if for every $\vec{x}_0$ we have a Green's function $G(\vec{x}, t|\vec{x}_0)$ that solves the initial value problem

$$\frac{\partial}{\partial t} G(\vec{x}, t|\vec{x}_0) = \mathcal{D} G(\vec{x}, t|\vec{x}_0), \qquad G(\vec{x}, 0|\vec{x}_0) = \delta(\vec{x} - \vec{x}_0) , \tag{6}$$

then we can construct a solution to equation (4) for any initial distribution ρ_0 via

$$\rho(\vec{x}, t) = \int G(\vec{x}, t|\vec{y}) \rho_0(\vec{y})\, \mathrm{d}^n y , \tag{7}$$

i.e. show that ρ defined by equation (7) fulfills $\rho(\vec{x}, 0) = \rho_0(\vec{x})$ for all $\vec{x}$ and solves equation (4).

Remark: The function $G(\vec{x},t|\vec{x}_0)$ is called the propagator of equation (4) because it propagates the initial distribution forward in time.

The rest of this problem will be devoted to finding the propagator $G(\vec{x},t|\vec{x}_0)$:

b) (3P) Take the spatial Fourier transform[1] of equation (6) to show that

$$\frac{\partial G(\vec{k},t|\vec{x}_0)}{\partial t} = k_i\Theta_{ij}\frac{\partial G(\vec{k},t|\vec{x}_0)}{\partial k^j} - k_iB_{ij}k_jG(\vec{k},t|\vec{x}_0)\,, \tag{8}$$

where $G(\vec{k},t|\vec{x}_0)$ is the spatial Fourier tranform of $G(\vec{x},t|\vec{x}_0)$.

Hint: You can assume that $G(\vec{k},t)\to 0$ as $|\vec{k}|\to\infty$.

c) To solve equation (8), we make the ansatz[2]

$$G(\vec{k},t|\vec{x}_0) = \exp\left[i\vec{\alpha}(t)^T\,\vec{k} - \frac{1}{2}\vec{k}^T\,\sigma(t)\,\vec{k}\right]\,, \tag{9}$$

where $\vec{\alpha}(t),\sigma(t)$ are a vector and a matrix valued function. With this ansatz we now need to find $\vec{\alpha}(t)$ and $\sigma(t)$:

1. (2P) Show that $\vec{\alpha},\sigma$ should fulfill

$$\frac{\partial\vec{\alpha}}{\partial t} = \Theta\vec{\alpha} \tag{10}$$

$$\frac{\partial\sigma}{\partial t} = \Theta\sigma + \sigma\Theta^T + 2B. \tag{11}$$

2. (2P) Which initial conditions do $\vec{\alpha}(t)$ and $\sigma(t)$ need to obey so that $G(\vec{x},0|\vec{x}_0)=\delta(\vec{x}-\vec{x}_0)$?

   Hint: Remember that $\int\exp(-i\vec{k}\cdot(\vec{x}-\vec{x}_0))\mathrm{d}^nk = (2\pi)^n\delta(\vec{x}-\vec{x}_0)$.

3. (1P) Write down the solution for equation (10) with the correct initial condition.

4. (1P) Show that

$$\sigma(t) = 2\int_0^t e^{(t-s)\Theta}Be^{(t-s)\Theta^T}\,\mathrm{d}s \tag{12}$$

   solves equation (11) and has the appropriate initial condition.

d) (2P) Use the inverse Fourier transform to show that our solution reads

$$G(\vec{x},t|\vec{x}_0) = \frac{1}{\sqrt{(2\pi)^n\det(\sigma)}}e^{-1/2(\vec{x}-\vec{\alpha}(t))^T\sigma^{-1}(t)(\vec{x}-\vec{\alpha}(t))}. \tag{13}$$

Hint: Remember problem 3 on sheet 3.

e) (1P) Use the results from problem 1 to show that the mean value and the mean squared deviation of our solution obey

$$\langle\vec{x}(t)\rangle = \vec{\alpha}(t), \tag{14}$$

$$\langle\vec{x}(t)\vec{x}(t)^T\rangle = \sigma(t) + \vec{\alpha}(t)\vec{\alpha}(t)^T. \tag{15}$$

f) (1P) Finally, consider the 1 dimensional special case $\Theta=0$, $B=D$ with the initial distribution $\rho_0(\vec{x}) = \delta(\vec{x})$. How does equation (4) look in this case? What are $\langle\vec{x}(t)\rangle$ and $\langle\vec{x}(t)\vec{x}(t)^T\rangle$ in this case?

[1] Remember that our convention for Fourier transforms yields in this case $G(\vec{k},t|\vec{x}_0) = \int e^{i\vec{k}\cdot\vec{x}}G(\vec{x},t|\vec{x}_0)\,\mathrm{d}^nx$ and $G(\vec{x},t|\vec{x}_0) = (2\pi)^{-n}\int e^{-i\vec{k}\cdot\vec{x}}G(\vec{k},t|\vec{x}_0)\,\mathrm{d}^nk$.

[2] The inspiration for this ansatz comes from a scheme for reducing PDEs to ODEs known as the ”method of characteristics”.