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Hand in on November 07th 2016 before the lecture

Advanced Statistical Physics II – Problem Sheet 3

Problem 1 – **Weiss-Curie ferromagnetism**

A ferromagnet consists of a system of spins s_i with $i = i, \ldots, N$, where each spin is interacting with every other spin. The Hamiltonian of the system is given by

$$H(s) = -\frac{J}{2N} \sum_i \sum_j s_i s_j - h \sum_i s_i \,, \tag{1}$$

where the spins can have the value $s_i = \pm 1$. Furthermore, J is the coupling constant between the spins and h is an external magnetic field. Here, we also allow for self-interaction of the spins, i.e. a term of the type s_i^2, which is included in this Hamiltonian.

a) (2P) In the zeroth problem sheet we showed that

$$\sqrt{2\pi}\, \mathrm{e}^{b^2/2} = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-x^2/2 - bx} \tag{2}$$

Use this to show that the partition function Z can be written as

$$Z(\beta, J, h) = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-N\beta J x^2/2} C(x) \,, \tag{3}$$

where

$$C(x) = \prod_i \sum_{\{s_i\}} \mathrm{e}^{\beta s_i (Jx + h)} \tag{4}$$

denotes the sum over all possible configurations and $1/\beta = k_B T$.
b) (1P) Calculate $C(x)$ explicitely.
c) (2P) Write the partition function as

$$Z(\beta, J, h) = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{N f(x)} \tag{5}$$

and show that $f(x)$ has an extremum results that is described by the equation

$$x = \tanh(\beta J x + \beta h) \,. \tag{6}$$

d) (2P) In the following we set $h = 0$, which corresponds to a system without external field. Expand the $\tanh$ for high temperatures T ,i.e. $\beta J \ll 1$, up to the second non-vanishing term. Argue that in this case there is only one solution to Eq. 6 and show that it is a maximum of $f(x)$.
e) (1P) Perform a saddle point approximation of the integral for Z around the maximum of $f(x)$ and solve the resulting integral. What is the free energy per particle F/N in this case?
f) (1P) We now analyze the limit $T \to 0$. Where are the extrema now and what kind of extrema are these?
Hint: Here it is enough to approximate $\tanh$ *by the first non-vanishing order, which can also be a constant.*
g) (1P) Perform a saddle point approximation of Z and calculate the partition functions for each maximum separately for the case $T \to 0$. Sum up all partition functions and calculate the free energy per particle from

the result.

Problem 2 – **Fluctuations in the Grand Canonical Ensemble**
In the lecture we derived the partition function $Z(T,V,N)$ for the canonical ensemble, where we held the particle number N and volume V constant. If we allow the system to exchange particles with the reservoir, we end up at the grand-canonical ensemble with its partition function

$$\Xi = \sum_{N=0}^{\infty} \sum_{i} e^{\beta\mu N - \beta U_i} = \sum_{N=0}^{\infty} e^{\beta\mu N} Z(T,V,N) \tag{7}$$

a) (1P) Show that the expecation value for N is given by

$$\langle N \rangle = k_B T \frac{\partial \ln \Xi}{\partial \mu} \tag{8}$$

b) (2P) Derive the number fluctuation $\langle \Delta N^2 \rangle \equiv \langle N^2 \rangle - \langle N \rangle^2$ from

$$k_B T \left(\frac{\partial \langle N \rangle}{\partial \mu} \right)_{T,V} . \tag{9}$$

c) (3P) Show that

$$\left(\frac{\partial \langle N \rangle}{\partial \mu} \right)_{T,V} = \frac{N^2}{V} \kappa_T, \quad \text{with} \quad \kappa_T = -\frac{1}{V} \left(\frac{\partial V}{\partial p} . \right)_{N,T} \tag{10}$$

Hint: Remember that $G = \mu N$

Problem 3 – **Multivariate Gaussian integral**

Consider the n-dimensional Gaussian integrals with arbitrary positive definite symmetric bilinear forms, i.e. a symmetric matrix $G = (g_{ij})$ with positive eigenvalues (λ_i).

a) (3P) Show that

$$\int d^n x \, e^{-\frac{1}{2}\vec{x}^T G \vec{x}} = \int d^n x \, e^{-\frac{1}{2} x_i g_{ij} x_j} = \frac{(2\pi)^{n/2}}{\sqrt{det(G)}} . \tag{11}$$

Hint: Use the spectral theorem.

f) (1P) With the result from a) solve the integral

$$\int d^n x \, e^{-\frac{1}{2}\vec{x}^T G \vec{x} - \vec{b} \cdot \vec{x}} . \tag{12}$$