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Hand in on November 28th 2016 before the lecture

# Advanced Statistical Physics II – Problem Sheet 6

Problem 1 – **Harmonic oscillator**

Consider the harmonic oscillator with the Hamiltonian $H(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^{3N} \frac{p_i^2}{2m} + \sum_{i=1}^{3N} \frac{1}{2} m\omega^2 q_i^2$.

a) Calculate the phase space density $\rho(\mathbf{q}, \mathbf{p})$ with initial condition $\mathbf{q}_0$ and $\mathbf{p}_0$.

b) Show explicitly from your result in a) that $\frac{\partial \rho}{\partial t} + v_j \nabla_j \rho = 0$, where $\mathbf{v} = (\dot{q}_1, \dots, \dot{q}_N, \dot{p}_1, \dots, \dot{p}_N)$ and $\nabla = (\partial_{q_1}, \dots, \partial_{q_N}, \partial_{p_1}, \dots, \partial_{p_N})$

c) Consider the one-dimensional case with given total energy of $H(q,p) = E = const$. How long does it take to visit every point in phase space? Sketch the phase space trajectory.

Problem 2 –**Liouville operator**

a) Derive the adjoint $L^\dagger$ of the Liouville operator $L$. Recall that the definition of the adjoint is given by

$$\langle A, L\rho \rangle = \langle L^\dagger A, \rho \rangle, \tag{1}$$

where $\langle X, Y \rangle = \int dq^{3N} dp^{3N} XY$ is the scalar product in phase space. Is the Liouville operator self-adjoint, not self-adjoint, or anti self-adjoint?

b) Derive an expression for the adjoint of $L^n$.

Problem 3 – **Liouville equation**

a) Prove

$$\int dp dq \rho(\tilde{q}, \tilde{p}, t+\tau; q, p, t) = \rho(\tilde{q}, \tilde{p}, t+\tau), \tag{2}$$

where $\rho(\tilde{q}, \tilde{p}, t+\tau; q, p, t)$ is the joint probability distribution resulting in the probability $\rho dp dq d\tilde{q} d\tilde{p}$ of finding the system at $\tilde{q}, \tilde{p}$ at time $t+\tau$, and at $q, p$ at time $t$. For this use Bayes' theorem (relation for conditional probabilities) and the Liouville propagators $e^{-\tau L}$ and $e^{-tL}$.

b) Show that if $\rho_0(q,p)$ is a stationary distribution then

$$e^{-tL(q,p)} \rho_0(q,p) = \rho_0(q,p), \tag{3}$$

where $L(q,p)$ is the Liouville operator.

Problem 4 – **Green's functions**

The concept of Green's functions is a useful tool to solve inhomogeneous linear partial differential equations. Consider the general form

$$Ly(\mathbf{x}) = f(\mathbf{x}), \tag{4}$$

where $L$ is a linear differential operator, $y(\mathbf{x})$ is a function that one wants to solve for, and $f(\mathbf{x})$ is an inhomogeneity. The function that satisfies $LG = \delta(\mathbf{x} - \mathbf{x}_0)$ is called the Green's function. The general solution of Eq. (4) is then given by

$$Ly = f = \delta \star f = (LG) \star f = L(G \star f) \tag{5}$$

$$y = G \star f = \int dx' G(x - x') f. \tag{6}$$

The star operator $\star$ describes the convolution operator. We want to use this concept to find a solution to the Poisson equation $\Delta\Phi = \rho(\mathbf{r})$ in 3D.

a) Write down the equation that defines the Green's function $G(\mathbf{r})$.

b) Fourier transform the Eq. in a) and derive an expression for the Fourier transform of Green's function $\tilde{G}(\mathbf{k})$.
c) Back-transform $\tilde{G}(\mathbf{k})$ to $G(\mathbf{r})$ to obtain the Green's function in real space and write down the formal solution of the Poisson equation.
*Hint: For the back-transform use spherical coordinates for* $\mathbf{k}$ *and* $\mathbf{r}$

d) Now consider the differential operator $L = \partial_t^2 + 2\gamma\partial_t + \omega_0^2$ of the damped 1D harmonic oscillator. Derive its Green's function $G(t)$ that solves $LG(t) = \delta(t)$.