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Advanced Statistical Physics II – Problem Sheet 2

Problem 1 – **Saddle-point approximation**

Consider the integral:

$$I = \int_{-\infty}^{\infty} e^{\lambda f(x)} dx \tag{1}$$

where f is some real function and $\lambda > 0$ is large. Let x_0 be the location of the global maximum. Expanding $\lambda f(x)$ around x_0 we obtain:

$$\lambda f(x) \simeq \lambda f(x_0) + \frac{1}{2}\lambda f''(x_0)(x - x_0)^2 + \mathcal{O}(x^3) \tag{2}$$

$$\Rightarrow \quad I \simeq e^{\lambda f(x_0)} \int_{-\infty}^{\infty} e^{\frac{\lambda}{2} f''(x_0)(x-x_0)^2} dx \tag{3}$$

(a) (2P) The canonical partition function Z is given by:

$$Z = \int dU e^{-F(U)/k_B T} \tag{4}$$

Consider the following free energy

$$F(U) = F_0 + bU + \frac{aU^2}{C_V T} \tag{5}$$

and calculate lnZ exactly.

(b) (2P) Convince yourself, that you get the *same* result using the saddle point approximation.

Problem 2 – **Canonical ensemble**

Recall that the free energy is given by:

$$F(T, V, N) = -k_B T lnZ(T, V, N) \tag{6}$$

(a) (2P) From equation (6), calculate

$$\frac{\partial F(T, V, N)}{\partial T} \tag{7}$$

(b) (2P) Convince yourself that

$$\frac{\partial \beta F(T,V,N)}{\partial \beta} = U \tag{8}$$

using only thermodynamics. ($\beta := \frac{1}{k_B T}$)

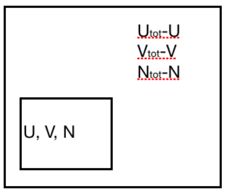
(c) (2P) Starting from the partition function Z, derive the *free energy* F for an ideal gas:

$$Z = \int dq^{3N} \int dp^{3N} \frac{e^{-\beta \mathcal{H}(q,p)}}{N! h^{3N}}, \quad \mathcal{H}(p,q) = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m}, \quad \int dq^{3N} = V^N \tag{9}$$

For this exercise, recall Stirling's approximation:

$$N! \simeq \sqrt{2\pi N} \left(\frac{N}{e}\right)^N \tag{10}$$

Problem 3 – **Particle number fluctuation**



Consider a system with both fluctuating energy and number of particles. The grand potential is given by:

$$\Omega(V,T,\mu) = -k_B T ln \Theta(V,T,\mu) \tag{11}$$

where $\Theta(V,T,\mu) = \sum_N \sum_i e^{-\beta(U_i - N\mu)}$

(a) (2P) Convince yourself that $< N >= -\frac{\partial \Omega}{\partial \mu}\Big|_{V,T}$

(b) (2P) Calculate $k_B T \frac{\partial < N >}{\partial \mu}\Big|_{V,T}$

(c) (2P) Calculate $\frac{\partial N}{\partial P}\Big|_{T,V}$ using $dV(N,P,T)$ and extensive / intensive arguments.

(d) (2P) Calculate $\frac{\partial N}{\partial \mu}\Big|_{T,V}$

Hint: Recall the Gibbs-Duhem Equation and the solution for (c)

(f) (2P) In the limit $N \to \infty$, find the scaling of the relative particle number deviation defined as

$$\frac{\sqrt{< N^2 > - < N >^2}}{< N >} \tag{12}$$