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Advanced Statistical Physics II – Problem Sheet 6

Problem 1 – **Principal value**

(2P) Derive the Fourier transform of the Heavyside step function $\theta(t)$.

Hint: Express $\theta(t)$ in form of the signum function $\text{sign}(t)$ and use the regularisation technique presented in the lecture.

Problem 2 – **Kramers-Kronig relations**

a) (2P) Given the even, $\chi_e(t)$, and odd, $\chi_o(t)$, parts of the response function $\chi(t)$ and their relation

$$\chi_o(t) = \text{sign}(t)\chi_e(t), \tag{1}$$

write $\chi(t)$ only in terms of $\chi_e(t)$ and derive a version of the Kramers-Kronig relation which expresses $\tilde{\chi}''(\omega)$ in terms of $\tilde{\chi}'(\omega)$.

b) (3P) Validate the Kramers-Kronig relation

$$\tilde{\chi}'(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\tilde{\chi}''(\omega')}{\omega' - \omega} \tag{2}$$

for the response function

$$\chi(t) = \theta(t)\sin(\bar{\omega}t). \tag{3}$$

c) (4P) Derive further alternative forms of the Kramers-Kronig relation

$$\tilde{\chi}'(\omega) = \frac{2}{\pi}\mathcal{P}\int_0^{\infty} d\omega' \frac{\omega'\tilde{\chi}''(\omega')}{\omega'^2 - \omega^2}, \tag{4}$$

$$\tilde{\chi}''(\omega) = \frac{2\omega}{\pi}\mathcal{P}\int_0^{\infty} d\omega' \frac{\tilde{\chi}'(\omega')}{\omega^2 - \omega'^2}. \tag{5}$$

Problem 3 – **Dielectric spectrum of water**

Consider the rotational equation of motion of a water molecule:

$$I\ddot{\phi}(t) + \gamma_R \dot{\phi}(t) + k\phi(t) = M(t) \tag{6}$$

where I is the moment of inertia, γ_R is the rotational friction, k is a "spring constant" $M(t)$ is an external torque.

a) (1P) Compute the response function $\tilde{\chi}(\omega)$ to the external torque $\tilde{M}(\omega)$.

b) (2P) Compare your expression of $\tilde{\chi}(\omega)$ with the Debye form $\tilde{\chi}_D(\omega)$

$$\tilde{\chi}_D(\omega) = (k - i\omega\gamma_R)^{-1} \tag{7}$$

c) (2P) Compute the resonance frequency of the response function $\tilde{\chi}_D(\omega)$.

d) (2P) Compute $\chi_D(t)$.

   Hint: Remember problem set 0.

e) Compute the energy uptake per period for an external torque $M(t) = A_0 \cos(\bar{\omega}t)$ for a system obeying the response χ_D.

$$P = \bar{\omega}/2\pi \int_0^{2\pi/\bar{\omega}} dt\ M(t)\dot{\phi}(t) \tag{8}$$

   Notice: The points for problem 3 e) were removed since there was a mistake in the previous version.