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Advanced Statistical Physics II – Problem Sheet 5

Problem 1 – **Ornstein-Uhlenbeck process**

Consider the Ornstein-Uhlenbeck process, which is a slight generalization of the Langevin equation introduced in the lecture,

$$m\dot{v}(t) = -\gamma[v(t) - \bar{v}] + F_R(t), \tag{1}$$

where m is the mass, γ is the friction constant, $\bar{v}$ is an additional constant velocity and $F_R(t)$ is the random force which fulfills $\langle F_R(t)\rangle = 0$ and $\langle F_R(t)F_R(t')\rangle = 2\gamma k_B T\delta(t - t')$.

a) (3P) Derive the solution for the above equation

$$v(t) = v(0)\mathrm{e}^{-\gamma t/m} + \bar{v}\left[1 - \mathrm{e}^{-\gamma t/m}\right] + \frac{1}{m}\int_0^t \mathrm{d}t'\mathrm{e}^{-\gamma(t-t')/m}F_R(t'), \tag{2}$$

using the same method of variation of the constant as in the lecture.

b) (2P) By averarging over the random force find the average velocity $\langle v(t)\rangle$ and its behaviour in the short $\langle v(0)\rangle$ and long time limits $\langle v(\infty)\rangle$.

c) (2P) Derive the velocity autocorrelation $\langle v(0)v(t)\rangle$ and its long time limit $\langle v(0)v(\infty)\rangle$. Use that $\langle v^2(0)\rangle = k_B T/m$.

d) (3P) Derive the equal time velocity correlation $\langle v^2(t)\rangle$ and its long time limit $\langle v^2(\infty)\rangle$.

Comment 1: The kinetic energy at $t = \infty$ contradicts the equipartition theorem. Does the term $\gamma\bar{v}$ in the equation of motion follow from Hamilton's equation?

Comment 2: The Ornstein-Uhlenbeck process is often used in financial modelling, an example is the Vasicek model. Can you understand its advantages for this application from the previous results?

Problem 2 – **Mean squared displacement**

a) (6P) Now use eq. (2) to calculate the mean squared displacement $\mathrm{MSD}(t) = \langle(x(t)-x(0))^2\rangle = \langle\int_0^t d\tau \int_0^t d\tau' v(\tau)v(\tau')\rangle$.

   Hint: Remember that $\int_0^t dt' \int_0^{t'} dt'' f(t'', t') = \int_0^t dt'' \int_{t''}^t dt' f(t'', t')$

b) (2P) What is the scaling of the mean squared displacement $\mathrm{MSD}(t)$ in the long time limit $t \to \infty$? In constrast, what is the scaling in the case of $\bar{v} = 0$.

c) (2P) Expand the exponential to second order in t to obtain the short time limit of the mean squared displacement $\mathrm{MSD}(t)$.