# Multiple surface wave solutions on linear viscoelastic media

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**Abstract** – In this supplement, we first derive the implicit dispersion relation, eq. (4) in the main text. We then show how the dispersion relations of Rayleigh, capillary-gravity and Lucassen waves follow, and derive the CGV wave dispersion relation, eq. (15) in the main text. Finally, we supply a wave existence state diagram for the surface tension $\sigma_{2\mathrm{D}} = 10$ mN/m, and give an example for the frequency dependent crossover from capillary-gravity and Lucassen waves to Rayleigh waves.

**Derivation of the conditional equation.** –

*Linear viscoelasticity.* For the bulk medium at $x_3 < 0$, the linearized continuum mechanical momentum conservation equations are given as [1]

$$\rho(\vec{x},t)\partial_t^2 u_j(\vec{x},t) = \partial_k \sigma_{jk}(\vec{x},t) + F_j(\vec{x},t) \qquad j \in \{1,2,3\}, \tag{S1}$$

where $\rho(\vec{x},t)$ is the mass density, $\vec{u}(x,t)$ is the displacement field and $\vec{F}(x,t)$ an external force. We assume the bulk medium to be a linear, isotropic and homogeneous viscoelastic medium, so that the dependence of the stress tensor $\sigma_{jk}$ on the displacement field is given by the viscoelastic stress strain relation [1]

$$\begin{aligned}\sigma_{jk}(\vec{x},t) = &\int_{-\infty}^{\infty} g_\mathrm{s}(t-t')\partial_{t'}\epsilon_{jk}(\vec{x},t)\ \mathrm{d}t' \\ &+ \frac{\delta_{jk}}{3}\int_{-\infty}^{\infty}\left[g_\mathrm{d}(t-t') - g_\mathrm{s}(t-t')\right]\partial_{t'}\epsilon_{ll}(x,t')\ \mathrm{d}t',\end{aligned} \tag{S2}$$

where the components of the strain tensor are given by

$$\epsilon_{jk} = \frac{1}{2}\left(\partial_j u_k + \partial_k u_j\right), \tag{S3}$$

and the shear and dilational relaxation functions $g_\mathrm{s}(t)$, $g_\mathrm{d}(t)$ are independent of position. We use the Einstein summation convention, so that $\epsilon_{ll} = \partial_l u_l = \vec{\nabla}\cdot\vec{u}$. Furthermore, for the temporal Fourier transform $\tilde{f}(\omega)$ of a function $f(t)$, we use the convention

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} f(t)\ \mathrm{d}t, \tag{S4}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\tilde{f}(\omega)\ \mathrm{d}\omega. \tag{S5}$$

The temporal Fourier transform of eq. (S2) then reads

$$\begin{aligned}\tilde{\sigma}_{jk}(\vec{x},\omega) = &(-i\omega)\tilde{g}_\mathrm{s}(\omega)\tilde{\epsilon}_{jk}(\vec{x},\omega) \\ &+ \delta_{jk}\frac{-i\omega}{3}\left[\tilde{g}_\mathrm{d}(\omega) - \tilde{g}_\mathrm{s}(\omega)\right]\tilde{\epsilon}_{ll}(\vec{x},\omega)\end{aligned} \tag{S6}$$

and the Fourier transformed relaxation functions $\tilde{g}_\mathrm{s}(\omega)$, $\tilde{g}_\mathrm{d}(\omega)$, are related to the, in general frequency dependent and complex, shear and bulk moduli $\mu$, $K$ [2] via

$$\mu(\omega) = \frac{-i\omega}{2}\tilde{g}_\mathrm{s}(\omega), \qquad K(\omega) = \frac{-i\omega}{3}\tilde{g}_\mathrm{d}(\omega). \tag{S7}$$

In elasticity theory, $\mu(\omega)$ and $K(\omega)$ are purely real and independent of $\omega$.

*Compressible Newtonian fluid.* Since we want to apply our theory to situations where water is the bulk medium, we need to explain how the stress strain relation (S2) is related to the usual stress strain relation of a compressible Newtonian fluid,

$$\sigma_{jk} = -P\delta_{jk} + 2\eta\partial_t\epsilon_{jk} + \left(\eta' - \frac{2}{3}\eta\right)\delta_{jk}\partial_t\epsilon_{ll}, \tag{S8}$$

where $\eta$, $\eta'$ are the shear and volume viscosities and $P$ is the pressure. We do this by including gravity and compressibility effects and employing an argumentation usually carried out to derive sound waves in bulk media, see e.g. refs [3,4]: Gravitational acceleration is modeled as an external force $F_i = -\delta_{i3}\rho g$, and we are looking for solutions to the linearized momentum conservation eq. (S1) with the stress tensor given by eq. (S8).

We first recall the standard stationary solution of eq.

(S1) for an incompressible ($\vec{\nabla} \cdot \vec{u} = \epsilon_{ll} = 0$) fluid at rest,

$$\vec{u}^{(0)}(\vec{x},t) = \vec{0}, \tag{S9}$$

$$\rho^{(0)}(\vec{x},t) = \rho_0, \tag{S10}$$

$$P^{(0)}(\vec{x},t) = P_0 - \rho_0 g x_3, \tag{S11}$$

where $\rho_0$ is the constant density and $P_0$ is the pressure at $x_3 = 0$. We will consider small perturbations around this solution, and by using this incompressible stationary solution as reference state, we assume that for the depths where the small perturbations are not negligibly small, the compression of the Newtonian fluid due to gravity can be neglected. To quantify this statement, we first recall the thermodynamic definition of the modulus of compression (bulk modulus),

$$\frac{1}{K} = -\frac{1}{V}\frac{\partial V}{\partial P}. \tag{S12}$$

The value of $K$ will in general depend not only on the material, but also on the thermodynamics of the process one is interested in: For adiabatic compression, $K \equiv K_S = 1/\kappa_S$ with $\kappa_S$ the adiabatic compressibility, while for isothermal compression, $K \equiv K_T = 1/\kappa_T$, with $\kappa_T$ the isothermal compressibility. For the equilibrium solution, the isothermal bulk modulus is appropriate, and for water at 20 °C it has the value [5]

$$K_T \approx 2.2 \times 10^8 \text{ kg/m}^2. \tag{S13}$$

The assumption that gravitational compression in the steady state solution can be neglected reads

$$\frac{\rho_0 g L}{K_T} \ll 1, \tag{S14}$$

where $L$ is the depth where the perturbation is so small that it can be neglected. Thus, for $\rho_0 = 10^3$ kg/m$^3$, $g = 9.81$ m/s$^2$,

$$L \ll 2.3 \times 10^4 \text{ m}. \tag{S15}$$

With the exception of low frequency gravity waves, the waves we consider decay at depths in the range of meters or less, so that neglecting the compression of the equilibrium state due to gravity is justified. Nevertheless, the following derivation could also be carried out to first order in $\rho_0 g x_3/K_T$. It would, however, be more complicated and lead to the same boundary condition, so that we perturb around the incompressible steady state solution (S9-S11).

We now consider small perturbations around the steady state solution of the form

$$\vec{u}(\vec{x},t) = \vec{0} + \vec{u}^{(1)}(\vec{x},t), \tag{S16}$$

$$\rho(\vec{x},t) = \rho_0 + \rho^{(1)}(\vec{x},t), \tag{S17}$$

$$P(\vec{x},t) = P_0 - \rho_0 g\left(x_3 + u_3^{(1)}(\vec{x},t)\right) + P^{(1)}(\vec{x},t), \tag{S18}$$

where $u_3^{(1)}$ is the $\vec{e}_3$-component of $\vec{u}^{(1)}$.

Momentum conservation, eq. (S1), constitutes only 3 equations for the 5 unknowns $\rho^{(1)}$, $P^{(1)}$, $\vec{u}^{(1)}$. To close the system of equations, we introduce linearized mass conservation,

$$0 = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \partial_t \vec{u}), \tag{S19}$$

and the definition of the modulus of compression (bulk modulus),

$$\frac{1}{K} = -\frac{1}{V}\frac{\partial V}{\partial P^{(1)}}, \tag{S20}$$

which gives a relation between local volume changes and local pressure changes. Technically, the bulk modulus we defined earlier and denoted by the same symbol, eq. (S12), is the physical bulk modulus, but because we neglect the compression due to gravity we use eq. (S20), where $P$ has been replaced by $P^{(1)}$, so that the gravitational contribution (the second term in eq. (S18)) has been excluded.

Integrating eq. (S19) with respect to time leads to

$$\rho^{(1)} = -\rho_0 \vec{\nabla} \cdot \vec{u}^{(1)}, \tag{S21}$$

where the constant of integration has been chosen so that for a volume preserving perturbation, $\vec{\nabla} \cdot \vec{u}^{(1)} = 0$, we obtain vanishing density change, $\rho^{(1)} = 0$. Furthermore, eq. (S20) can be integrated to yield, to first order in $\vec{u}^{(1)}$,

$$P^{(1)} = -K \vec{\nabla} \cdot \vec{u}^{(1)}, \tag{S22}$$

where we used the relation $\Delta V/V \approx \vec{\nabla} \cdot \vec{u}^{(1)}$ [2] and chose the constant of integration so that for a volume preserving perturbation, $\vec{\nabla} \cdot \vec{u}^{(1)} = 0$, the pressure change vanishes, $P^{(1)} = 0$.

Plugging the perturbed solution, eqs. (S16-S18), into momentum conservation, eq. (S1), and using eq. (S21), (S22) to eliminate $P^{(1)}$, $\rho^{(1)}$, we obtain, to linear order in the perturbations,

$$\begin{aligned}\rho_0 \partial_t^2 u_j^{(1)} = {} & \partial_k \left[2\eta \partial_t \epsilon_{jk}^{(1)} + \left(\eta' - \frac{2}{3}\eta\right) \delta_{jk} \partial_t \epsilon_{ll}^{(1)}\right] \\ & + \partial_j \left[-\rho_0 g u_3^{(1)} + K \vec{\nabla} \cdot \vec{u}^{(1)}\right] + \delta_{j3} g \rho_0 \vec{\nabla} \cdot \vec{u}^{(1)},\end{aligned} \tag{S23}$$

where $\epsilon_{jk}^{(1)}$ is the strain tensor, eq. (S3), with $\vec{u}$ replaced by $\vec{u}^{(1)}$. Gravitational acceleration enters this equation via two terms: The restoring gravitational force (first term in second line of eq. (S23)), and the buoyancy (last term of eq. (S23)). One standard way to deal with these terms is the surface gravity approximation, where gravitational force terms are neglected in the bulk, but the restoring gravitational force is kept in the boundary condition [6]. In that case, eq. (S23) is simplified in the bulk, to yield

$$\rho_0 \partial_t^2 u_j^{(1)} = \partial_k \left[2\eta \partial_t \epsilon_{jk}^{(1)} + \left(\eta' - \frac{2}{3}\eta\right) \delta_{jk} \partial_t \epsilon_{ll}^{(1)} + \delta_{jk} K \epsilon_{ll}^{(1)}\right]. \tag{S24}$$

Comparing this to eqs. (S1), (S6), we see that our description of a compressible Newtonian fluid in the presence of gravity has now been approximated by that of a viscoelastic medium without external forces and with relaxation functions

$$\tilde{g}_{\mathrm{s}}(\omega) = 2\eta, \tag{S25}$$

$$\tilde{g}_{\mathrm{d}}(\omega) = 3\eta' + \frac{3K}{-i\omega}. \tag{S26}$$

The use of the symbol $K$ here is consistent with our earlier use of the symbol in eq. (S7): If the dilational response is purely elastic, then $\eta' = 0$, and it can be seen by direct comparison that the two formulas for $\tilde{g}_{\mathrm{d}}$, eqs. (S7) and (S26), coincide if $K(\omega)$ in eq. (S7) is real and independent of $\omega$. Formally, the relaxation function (S26) describes a Kelvin-Voigt material [1], where for low frequencies the bulk modulus $K$ dominates while for high frequencies the volume viscosity $\eta'$ dominates. If we assume that the compression due to the perturbation is adiabatic, we can use the relation

$$K \equiv K_S = \rho_0 c_{bulk}^2 \tag{S27}$$

to calculate the adiabatic bulk modulus $K_S$ from the long wavelength sound velocity $c_{bulk}$ [3]. For water at 25 °C, the relevant parameters are given by [5]

$$\eta \approx 1 \times 10^{-3}\ \mathrm{Pa \cdot s}, \tag{S28}$$

$$\eta' \approx 3 \times 10^{-3}\ \mathrm{Pa \cdot s}, \tag{S29}$$

$$\rho_0 \approx 1 \times 10^{3}\ \frac{\mathrm{kg}}{\mathrm{m}^3}, \tag{S30}$$

$$c_{bulk} \approx 1.5 \times 10^{3}\ \frac{\mathrm{m}}{\mathrm{s}}. \tag{S31}$$

For these values and frequencies $\omega \ll 10^{12}\ \mathrm{s}^{-1}$, we have $|\eta'| \ll \left|\rho_0 c_{bulk}^2/(-i\omega)\right|$, so that the response of water can be thought of as purely viscous in shear and purely elastic in dilation. We remark that at sufficiently high frequencies, bulk water also shows an elastic response to shear deformation [7, 8]. These effects, which start to become important for frequencies in the GHz regime, could be incorporated in our model by replacing the shear relaxation function $\tilde{g}_{\mathrm{s}}$ by the relaxation function of a Maxwell material, i.e.

$$\tilde{g}_{\mathrm{s}}(\omega) = \frac{2\eta}{1 - i\omega\tau}, \tag{S32}$$

where $\tau = \eta/\mu$ is a time scale characteristic for the material and $\mu$ characterizes the shear elasticity. In the limit $\tau \to 0$, the elasticity is negligible and one recovers eq. (S25).

Comparing eq. (S23) in the surface gravity approximation with the momentum conservation eq. (S1), the temporal Fourier transform of the stress tensor at the interface $x_3 = 0$ becomes

$$\begin{aligned}\tilde{\sigma}_{jk} = &-\delta_{jk}\left[\delta(\omega)P_0 - g\rho_0\tilde{u}_3^{(1)}\right] \\ &+ (-i\omega)\tilde{g}_{\mathrm{s}}\tilde{\epsilon}_{jk}^{(1)} + \delta_{jk}\frac{-i\omega}{3}\left(\tilde{g}_{\mathrm{d}} - \tilde{g}_{\mathrm{s}}\right)\tilde{\epsilon}_{ll}^{(1)},\end{aligned} \tag{S33}$$

where the displacement field $\tilde{u}_j^{(1)}$ and its derivatives are assumed to be evaluated at $x_3 = 0$. Equation (S33) differs from the viscoelastic stress strain relation, eq. (S6) by the first term, which includes a homogeneous background pressure and the restoring gravitational force at the surface. Although these terms do not appear in the approximate momentum conservation eq. (S24), they will appear in stress boundary conditions. If gravity can be neglected, $g = 0$, and the steady state pressure at the interface is set to zero, $P_0 = 0$, eq. (S33) simplifies to eq. (S6).

*The stress boundary conditions for a viscoelastic interface.* A detailed derivation of the continuum mechanical boundary conditions of two bulk media divided by a viscoelastic surface was given by Kralchevsky et. al. [9]. In plane, they assumed the surface to have a purely viscous shear response with viscosity $\eta_{2\mathrm{D}}$, a viscoelastic response under dilation with viscosity $\eta'_{2\mathrm{D}}$ and a position dependent surface tension $\sigma_{2\mathrm{D}}$. For out of plane deformations, they assumed a bending rigidity $\kappa_{2D}$ and a transverse viscosity $\eta^{\perp}_{2\mathrm{D}}$. Furthermore, they considered the interface to have an area mass density $\rho_{2\mathrm{D}}$ and included an external force per area $\vec{f}_s$ acting on the surface. To linear order, they obtained [1]

$$\begin{aligned}\rho_{2\mathrm{D}}\partial_t v_{2\mathrm{D},\alpha} = &\left(\sigma_{\mathrm{III},n\alpha} - \sigma_{n\alpha}\right) + \rho_{2\mathrm{D}} f_{s,\alpha} + \left(\vec{\nabla}_s\right)_\alpha \sigma_{2\mathrm{D}} \\ &+ \eta'_{2\mathrm{D}}\left(\vec{\nabla}_s\right)_\alpha\left(\vec{\nabla}_s\cdot\vec{v}_{2\mathrm{D}}\right) \\ &+ \eta_{2\mathrm{D}}\vec{\nabla}_s^2 v_{2\mathrm{D},\alpha} \qquad \alpha \in \{1,2\},\end{aligned} \tag{S34}$$

$$\begin{aligned}\rho_{2\mathrm{D}}\partial_t v_{2\mathrm{D},n} = &\left(\sigma_{\mathrm{III},nn} - \sigma_{nn}\right) + \rho_{2\mathrm{D}} f_{s,n} + \sigma_{2\mathrm{D}}\vec{\nabla}_s^2 u_{2\mathrm{D},3} \\ &- \kappa_{2D}\vec{\nabla}_s^2\vec{\nabla}_s^2 u_{2\mathrm{D},3} + \eta^{\perp}_{2\mathrm{D}}\vec{\nabla}_s^2\partial_t u_{2\mathrm{D},3},\end{aligned} \tag{S35}$$

where $u_{2\mathrm{D},3} \equiv u_{2\mathrm{D},3}(x_1, x_2, t)$ is the displacement of the surface point $(x_1, x_2, 0)$ in $x_3$-direction, $\vec{v}_{2\mathrm{D}}$ is the velocity at the surface, the stress tensors for the bulk media below and above the surface, $\sigma$, $\sigma_{\mathrm{III}}$, are understood to be evaluated at $x_3 = 0$ and $\vec{\nabla}_s := \vec{\nabla} - \hat{n}\left(\hat{n}\cdot\vec{\nabla}\right)$ is the projection of the gradient onto the surface, with $\hat{n}$ the unit normal vector pointing into the $x_3 > 0$ half space. The indices $\alpha$, $n$ label tensor components parallel and perpendicular to the surface, respectively. More generally, we will use the convention that greek indices run over $\{1, 2\}$, while latin indices will run over $\{1, 2, 3\}$. For the half-space $x_3 > 0$, we will not consider any dynamics and assume that there is a constant pressure $P_{\mathrm{III}}$, so that

$$\sigma_{\mathrm{III},jk} = -P_{\mathrm{III}}\delta_{jk}. \tag{S36}$$

To rewrite the boundary conditions (S34), (S35), in a form more useful for the present context, we will now relate the surface velocity to the surface displacement, explicitly evaluate $\vec{\nabla}_s$, relate the elastic in-surface dilational

[1]This is ( [9],5.16), ( [9],5.17), but with the membrane inertia and body force kept. Also in eq. ( [9],5.16), the stress tensor of the medium above the membrane was neglected, which we included.

response $\sigma_{2D}$ to the displacement and use gravitation as external force:

In the small displacement limit we approximate the velocity by the time derivative of the displacement,

$$\vec{v}_{2D} \approx \partial_t \vec{u}_{2D}, \tag{S37}$$

and the normal vector by the unit vector in $x_3$-direction, $\hat{n} \approx \hat{e}_3$. With the latter approximation, we get

$$\vec{\nabla}_s = \hat{e}_1\partial_1 + \hat{e}_2\partial_2 \equiv \hat{e}_\beta\partial_\beta. \tag{S38}$$

To express $\sigma_{2D}$ in terms of the surface displacement, we use a two dimensional version of the argumentation used to derive eq. (S26): Assuming that the surface mass density and surface tension deviate only slightly from their (homogeneous in space and time) equilibrium values, we can write

$$\rho_{2D}(x_1, x_2, t) = \rho_{2D}^{(0)} + \rho_{2D}^{(1)}(x_1, x_2, t), \tag{S39}$$

$$\sigma_{2D}(x_1, x_2, t) = \sigma_{2D}^{(0)} + \sigma_{2D}^{(1)}(x_1, x_2, t), \tag{S40}$$

with $\rho_{2D}^{(1)}, \sigma_{2D}^{(1)}$ small compared to $\rho_{2D}^{(0)}, \sigma_{2D}^{(0)}$. To express $\rho_{2D}^{(1)}$ in terms of the surface displacement field, we integrate the linearized 2D mass conservation equation,

$$\partial_t\rho_{2D}^{(1)} + \rho_{2D}^{(0)}\partial_t\partial_\beta u_{2D,\beta} = 0, \tag{S41}$$

with respect to time to get

$$\rho_{2D}^{(1)} = -\rho_{2D}^{(0)}\partial_\beta u_{2D,\beta}. \tag{S42}$$

The constant of integration has been chosen so that no area change, $\partial_\beta u_{2D,\beta} = 0$, implies no density change, $\rho_{2D}^{(1)} = 0$. To express $\sigma_{2D}^{(1)}$ in terms of the surface displacement field, we start with the definition of the 2D dimensional in-plane area elastic modulus $K_{2D}$,

$$\frac{1}{K_{2D}} = \frac{1}{A}\frac{\partial A}{\partial \sigma_{2D}}, \tag{S43}$$

which relates local changes in the area $A$ of a surface element to the local surface tension $\sigma_{2D}$. Note that because the surface tension can be thought of as a negative surface pressure, there is no minus sign here, contrary to eq. (S20). As in the three dimensional case, $K_{2D}$ not only depends on the material the interface is made of but also on the thermodynamics of the process one wants to investigate (e.g. isothermal, adiabatic). Eq. (S43) can be integrated to yield, to first order

$$\sigma_{2D}^{(1)} = -K_{2D}\partial_\beta u_{2D,\beta}, \tag{S44}$$

where, analogous to the three dimensional situation, we used $\Delta A/A = \partial_\beta u_{2D,\beta}$, and assumed that no compression implies no surface tension change. Eqs. (S42), (S44), can now be used to eliminate $\rho_{2D}^{(1)}$, $\sigma_{2D}^{(1)}$ in eqs. (S39), (S40), so that

$$\rho_{2D} = \rho_{2D}^{(0)}\left(1 - \partial_\beta u_{2D,\beta}\right), \tag{S45}$$

$$\sigma_{2D} = \sigma_{2D}^{(0)} - K_{2D}\partial_\beta u_{2D,\beta}. \tag{S46}$$

The influence of gravity on the membrane yields an area force density

$$\rho_{2D}\vec{f}_s = -\rho_{2D}g\vec{e}_3 = -\rho_{2D}^{(0)}g\left(1 - \partial_\beta u_{2D,\beta}\right)\vec{e}_3, \tag{S47}$$

where we used eq. (S45).

For the equilibrium solution (around which we perturb), the surface should be at rest at $x_3 = 0$. Therefore, the constant area force density acting on the interface, $-\rho_{2D}^{(0)}g\vec{e}_3$, has to be accounted for by a pressure difference in the bulk media directly below and above the surface. This means that, in the equilibrium solution, the constant background pressure $P_0$ in the lower bulk medium (c.f. eq. (S18)) and the constant pressure $P_{\text{III}}$ in the upper bulk medium need to differ exactly by the pressure the surface exerts on the lower medium in equilibrium, i.e. $P_0 - P_{\text{III}} = \rho_{2D}^{(0)}g$, so that the surface remains at $x_3 = 0$.

Plugging our expressions for $\vec{v}_{2D}$, $\vec{\nabla}_s$, $\rho_{2D}$, $\sigma_{2D}$ and $\rho_{2D}\vec{f}_s$, namely eqs. (S37), (S38), (S45), (S46), (S47), into the boundary conditions (S34), (S35), they can be rewritten, to linear order, as

$$\begin{aligned}\rho_{2D}^{(0)}\partial_t^2 u_{2D,\alpha} = {}& (\sigma_{\text{III},3\alpha} - \sigma_{3\alpha}) + (K_{2D} + \eta'_{2D}\partial_t)\,\partial_\alpha\partial_\beta u_{2D,\beta} \\ & + \eta_{2D}\partial_t\partial_\beta^2 u_{2D,\alpha} \qquad \alpha \in \{1,2\},\end{aligned} \tag{S48}$$

$$\begin{aligned}\rho_{2D}^{(0)}\partial_t^2 u_{2D,3} = {}& (\sigma_{\text{III},33} - \sigma_{33}) - \rho_{2D}^{(0)}g\left(1 - \partial_\beta u_{2D,\beta}\right) \\ & + \left(\sigma_{2D}^{(0)} + \eta_{2D}^{\perp}\partial_t - \kappa_{2D}\partial_\beta^2\right)\partial_\beta^2 u_{2D,3}.\end{aligned} \tag{S49}$$

Kralchevksy et. al. [9] remark that in their eqs. (5.16), (5.17), where body forces are neglected, the variables $u_{2D,1}, u_{2D,2}$ and the variable $u_{2D,3}$ are decoupled. In our equations this is not the case: Gravitation causes a downwards force if the membrane is locally compressed, reflected by the term $\rho_{2D}^{(0)}\partial_\beta u_{2D,\beta}$ in eq. (S49).

*The harmonic wave ansatz.* We want to describe the displacement field $\vec{u}(\vec{x}, t)$ in the bulk medium via displacement potentials $\varphi(\vec{x}, t)$, $\vec{\psi}(\vec{x}, t)$ as

$$\vec{u} = \vec{\nabla}\varphi + \vec{\nabla} \times \vec{\psi}. \tag{S50}$$

If the temporal Fourier transforms of the displacement potentials satisfy the Helmholtz equations

$$\rho_0(-i\omega)\tilde{\varphi} = \frac{1}{3}\left[2\tilde{g}_s(\omega) + \tilde{g}_d(\omega)\right]\Delta\tilde{\varphi}, \tag{S51}$$

$$\rho_0(-i\omega)\tilde{\psi}_j = \frac{1}{2}\tilde{g}_s(\omega)\Delta\tilde{\psi}_j, \qquad j \in \{1,2,3\}, \tag{S52}$$

where $\Delta$ is the Laplace operator and the tilde signifies temporal Fourier transform, then the displacement field (S50) fulfills the linearized momentum conservation eq. (S1) for a linear, isotropic, homogeneous viscoelastic material with stress strain relation (S2) and without external forces, as is appropriate for the surface gravity approximation discussed following eq. (S23).

The harmonic wave ansatz [10] then consists of choosing the displacement potentials

$$\varphi(x_1, x_3, t) = \Phi\exp\left(\lambda_l x_3\right)\exp\left[i\left(kx_1 - \omega t\right)\right], \tag{S53}$$

$$\psi_j(x_1, x_3, t) = \Psi\exp\left(\lambda_t x_3\right)\exp\left[i\left(kx_1 - \omega t\right)\right]\delta_{j,2}, \tag{S54}$$

where $j \in \{1,2,3\}$ and we assume $\omega \in \mathbb{R}$ is a given parameter, while $k$, $\lambda_l$, $\lambda_t$, $\Phi$, $\Psi \in \mathbb{C}$ are to be obtained. The requirement that the waves be damped as $x_3 \to -\infty$ implies $\mathrm{Re}(\lambda_l)$, $\mathrm{Re}(\lambda_t) > 0$. Our choice $\omega \in \mathbb{R}$, $k \in \mathbb{C}$ means we consider plane wave solutions with frequency $\omega$, which are damped as they propagate along the $\vec{e}_1$-axis, and that we will later solve for $k(\omega)$.

Plugging the harmonic wave ansatz into eqs. (S51), (S52), yields

$$\lambda_l^2(k,\omega) = k^2 + \gamma^2(\omega), \tag{S55}$$

$$\lambda_t^2(k,\omega) = k^2 + \alpha^2(\omega), \tag{S56}$$

where we defined

$$\gamma^2(\omega) := \frac{3(-i\omega)\rho_0}{2\tilde{g}_\mathrm{s}(\omega) + \tilde{g}_\mathrm{d}(\omega)}, \tag{S57}$$

$$\alpha^2(\omega) := \frac{2(-i\omega)\rho_0}{\tilde{g}_\mathrm{s}(\omega)}. \tag{S58}$$

Equations (S55), (S56) and the requirements $\mathrm{Re}(\lambda_l)$, $\mathrm{Re}(\lambda_t) > 0$ then determine $\lambda_l, \lambda_t$ uniquely [2]. More generally, for a complex number $z = |z|e^{i\theta}$ with complex phase $arg(z) = \theta \in (-\pi, \pi)$, we define the symbol $\sqrt{z} := \sqrt{|z|}e^{i\theta/2}$ to denote the complex square root with positive real part.

*The implicit dispersion relation.* Our ansatz for the upper and lower half space contains 6 parameters: $k$, $\omega$, $\Phi$, $\Psi$, $\lambda_l$, $\lambda_t$. We assume $\omega$ is constant and given. Equations (S55), (S56) determine $\lambda_l$, $\lambda_t$ as functions of $k, \omega$. The stress boundary conditions (S48), (S49) at $x_3 = 0$ yield a homogeneous linear system of two eqs. for the two coefficients $\Phi$, $\Psi$. This system can be obtained explicitly by calculating $\tilde{\sigma}_{ij}$, $\tilde{u}_{\mathrm{2D},i} = \tilde{u}_i|_{x_3=0}$, for the displacement field (S50) and then plugging this into eqs. (S48), (S49) (for (S48), only the $\alpha = 1$ case is needed, since for $\alpha = 2$ the equation is fulfilled trivially, as can be seen after a short calculation). For the stress tensor of the bulk medium, we use the generalized form (S33), to include effects of compressibility and gravity as discussed in the derivation of eq. (S33). The result is

$$\begin{aligned} 0 = ik &\left[ i\omega\rho_\mathrm{2D}^{(0)} - k^2\tilde{g}_\mathrm{2D} - \lambda_l\tilde{g}_\mathrm{s} \right] \Phi \\ &+ \left[ \lambda_t(k^2\tilde{g}_\mathrm{2D} - i\omega\rho_\mathrm{2D}^{(0)}) + \frac{1}{2}\tilde{g}_\mathrm{s}(k^2+\lambda_t^2) \right] \Psi \end{aligned} \tag{S59}$$

$$\begin{aligned} 0 = &\left[ \lambda_l(\omega^2\rho_\mathrm{2D}^{(0)} - k^2\tilde{\Pi}_\mathrm{2D} - \rho_0 g) - k^2\rho_\mathrm{2D}^{(0)} g \right. \\ &\left. + \frac{i\omega}{2}\tilde{g}_\mathrm{s}(k^2+\lambda_t^2) \right] \Phi \\ &+ \left[ ik(\omega^2\rho_\mathrm{2D}^{(0)} - k^2\tilde{\Pi}_\mathrm{2D} - \lambda_t\rho_\mathrm{2D}^{(0)} g) \right. \\ &\left. - ik\rho_0 g - \omega k\lambda_t\tilde{g}_\mathrm{s} \right] \Psi, \end{aligned} \tag{S60}$$

[2] If either $\lambda_l^2$ or $\lambda_t^2$ is purely real and negative, both square roots have vanishing real part, but we do not accept such solutions: The wave would not be damped away from the interface.

where again $\lambda_l$, $\lambda_t$ are given by eqs. (S55), (S56), and where the response functions $\tilde{g}_\mathrm{2D}$, $\tilde{\Pi}_\mathrm{2D}$ are

$$\tilde{g}_\mathrm{2D}(\omega) := \eta_\mathrm{2D} + \eta'_\mathrm{2D} + K_\mathrm{2D}/(-i\omega), \tag{S61}$$

$$\tilde{\Pi}_\mathrm{2D}(k,\omega) := \sigma_\mathrm{2D}^{(0)} + (-i\omega)\eta_\mathrm{2D}^{\perp} + k^2\kappa_{2D}. \tag{S62}$$

The symbol $\tilde{g}_\mathrm{2D}$ was chosen because of the structural resemblance of eqs. (S26), (S61). Since the perturbations $\rho^{(1)}$, $\rho_\mathrm{2D}^{(1)}$, $\sigma_\mathrm{2D}^{(1)}$ do not appear explicitly in eqs. (S59-S62) anymore, we will drop the subscripts 0 and superscripts (0) of the corresponding equilibrium values $\rho_0$, $\rho_\mathrm{2D}^{(0)}$, $\sigma_\mathrm{2D}^{(0)}$ in the following, and have also dropped it in the main text. In order to have a propagating wave with nonzero amplitude, the system (S59), (S60) has to have a nontrivial solution. Thus, the implicit dispersion relation is obtained by setting the determinant of the $2 \times 2$ coefficient matrix for $\Phi$, $\Psi$, obtained from (S59), (S60), equal to zero. This yields

$$\begin{aligned} 0 = 4 &\left( k^2\tilde{\Pi}_\mathrm{2D} + \rho g - \omega^2\rho_\mathrm{2D} \right) \\ &\times \left[ \left(k^2\tilde{g}_\mathrm{2D} - i\omega\rho_\mathrm{2D}\right)\left(k^2 - \lambda_l\lambda_t\right) + i\omega\rho\lambda_l \right] \\ &+ 4\left(k^2\tilde{g}_\mathrm{2D} - i\omega\rho_\mathrm{2D}\right)\omega^2\rho\lambda_t \\ &+ \tilde{g}_\mathrm{s}\left[ i\omega\tilde{g}_\mathrm{s}\left(-4k^2\lambda_l\lambda_t + (k^2+\lambda_t^2)^2\right) \right. \\ &\left. + 2\rho_\mathrm{2D} g k^2\left(2\lambda_l\lambda_t - (k^2+\lambda_t^2)\right) \right], \end{aligned} \tag{S63}$$

and thus completes our derivation of eq. (4) from the main text.

Finding a solution $k(\omega)$ to eq. (S63) yields the dispersion relation of a surface wave, for which phase velocity and propagation distance can then be calculated via

$$c(\omega) = \frac{\omega}{\mathrm{Re}(k(\omega))}, \tag{S64}$$

$$\beta(\omega) = \frac{1}{\mathrm{Im}(k(\omega))}. \tag{S65}$$

**Limiting cases.** – In the following subsections, we will first discuss how the known dispersion relations for Rayleigh waves, capillary-gravity-flexural and Lucassen waves emerge from eq. (S63). Then we will derive the dispersion relation for the capillary-gravity-viscosity (CGV) surface wave, eq. (15) of the main text.

*Rayleigh waves.* Upon removing the effects related to the surface ($\rho_\mathrm{2D} = 0$, $\tilde{g}_\mathrm{2D} = 0$, $\tilde{\Pi}_\mathrm{2D} = 0$) and also gravity ($g = 0$), eq. (S63) becomes

$$4k^2\lambda_l\lambda_t = \left(k^2 + \lambda_t^2\right)^2, \tag{S66}$$

where $\lambda_l$, $\lambda_t$ are given by eqs. (S55), (S56). This is the classical Rayleigh conditional equation whose solutions lead to the known (viscoelastic) Rayleigh waves [10–12]. Analytic solutions for it can be found as follows [10]:

Squaring eq. (S66), one obtains

$$0 = 8\left(\frac{k}{\alpha(\omega)}\right)^6 + 8(2-\nu(\omega))\left(\frac{k}{\alpha(\omega)}\right)^4 + 8(1-\nu(\omega))\left(\frac{k}{\alpha(\omega)}\right)^2 + (1-\nu(\omega)), \qquad \text{(S67)}$$

where

$$\nu(\omega) = \frac{\tilde{g}_{\mathrm{d}}(\omega) - \tilde{g}_{\mathrm{s}}(\omega)}{\tilde{g}_{\mathrm{s}}(\omega) + 2\tilde{g}_{\mathrm{d}}(\omega)} \qquad \text{(S68)}$$

is Poisson's ratio. Equation (S67) is cubic in $(k/\alpha_{\mathrm{I}}(\omega))^2$, and thus Cardano's formula [13] can be used to obtain its three roots

$$-\left(\frac{k(\omega)}{\alpha(\omega)}\right)^2 = \frac{2-\nu(\omega)}{3} + \frac{1}{3}\sqrt[3]{h_1(\nu(\omega))}e^{-i\pi(1-2n)/3} + \frac{1-\nu(\omega)+\nu(\omega)^2}{3\sqrt[3]{h_1(\nu(\omega))}}e^{i\pi(1-2n)/3}, \qquad \text{(S69)}$$

where

$$h_1(\nu) = \frac{1}{16}\Big(-11 + 3\nu - 24\nu^2 + 16\nu^3 + 3\sqrt{3}\sqrt{-5+26\nu-37\nu^2+48\nu^3-32\nu^4}\Big) \qquad \text{(S70)}$$

and $n \in \{1,2,3\}$ labels the distinct solutions. Rayleigh derived this formula originally for isotropic homogeneous elastic media, but the approach also works for isotropic homogeneous viscoelastic media [11,12,14]. Since formula (S69) is a solution to the squared Rayleigh conditional eq. (S67), it is not clear whether it also solves the original Rayleigh conditional eq. (S66). In the elastic case, only one of the solutions of the squared equation also solves the original equation, while in the viscoelastic case, there are situations where two of the solutions of eq. (S67) also solve the original eq. (S66), see refs. [11, 12]. Recently, there has been a report on three solutions [15]. However, in that work a real wave number $k$ was assumed and $\omega(k)$ was obtained, which amounts to solving (S66) for $\omega$ or, equivalently, finding the inverse function of $k(\omega)$. As $k(\omega)$ need not be monotonic, the inverse can be non-unique so that this finding does not answer the still open question whether viscoelastic materials exist where all three solutions $k(\omega)$ of the squared conditional eq. (S67) also solve the original Rayleigh conditional eq. (S66). For the water-like viscoelastic material, which we discuss as an example in the main text, we found that only two solutions of the squared conditional equation also solve the original conditional equation.

In the incompressible limit, $\tilde{g}_{\mathrm{d}} \to \infty$, Poisson's ratio reduces to $\nu(\omega) \equiv 1/2$, so that the right hand side of eq. (S69) becomes independent of $\omega$. Consequently, we get

$$k(\omega) \propto \alpha(\omega), \qquad \text{(S71)}$$

and using the definition of $\alpha$, eq. (S58), we see that for a Newtonian fluid with shear relaxation function given by eq. (S25),

$$k(\omega) \propto \omega^{1/2}. \qquad \text{(S72)}$$

According to eqs. (S64), (S65), this means that $c(\omega) \propto \omega^{1/2}$, $\beta(\omega) \propto \omega^{-1/2}$. This is indeed what can be observed in Fig. 2 (a), (d) and Fig. S1 (a), (d), indicating that compressibility effects are not relevant for Rayleigh waves at the shown frequencies.

*Factorization, capillary-gravity and longitudinal capillary waves.* There are situations where the conditional eq. (S63) factorizes, as we shall show now: Assuming that the gravitational force on the surface can be neglected, i.e. that

$$\rho_{2\mathrm{D}}g \ll \omega|\tilde{g}_{\mathrm{s}}|, \qquad \rho_{2\mathrm{D}}g|k|^4 \ll \omega|\tilde{g}_{\mathrm{s}}||\lambda_t|^4, \qquad \text{(S73)}$$

eq. (S63) first simplifies to

$$0 = 4\left(k^2\tilde{\Pi}_{2\mathrm{D}} + \rho g - \omega^2\rho_{2\mathrm{D}}\right) \times \left[(k^2\tilde{g}_{\mathrm{II}} - i\omega\rho_{2\mathrm{D}})\left(-k^2 + \lambda_l\lambda_t\right) - i\omega\rho\lambda_l\right] + (-i\omega)\tilde{g}_{\mathrm{s}}\left[2\left(k^2\tilde{g}_{2\mathrm{D}} - i\omega\rho_{2\mathrm{D}}\right)\lambda_t\alpha^2 + \tilde{g}_{\mathrm{s}}\left(-4k^2\lambda_l\lambda_t + (k^2+\lambda_t^2)^2\right)\right]. \qquad \text{(S74)}$$

Furthermore assuming that

$$1 - \frac{k^2}{\lambda_l\lambda_t} \approx 1, \qquad \text{(S75)}$$

$$\frac{1}{\alpha^4}\left[-4k^2\lambda_l\lambda_t + \left(k^2+\lambda_t^2\right)^2\right] \approx 1, \qquad \text{(S76)}$$

eq. (S74) factorizes to

$$0 = \left[(k^2\tilde{\Pi}_{2\mathrm{D}} + \rho g - \omega^2\rho_{2\mathrm{D}})\lambda_l - \omega^2\rho\right] \times \left[\lambda_t(k^2\tilde{g}_{2\mathrm{D}} - i\omega\rho_{2\mathrm{D}}) - i\omega\rho\right] \qquad \text{(S77)}$$

If the approximations (S73), (S75), (S76) are justified can of course only be checked after one has found a solution to eq. (S63), since $k(\omega)$ appears in the conditions. We want to point out an example where the rather complicated looking conditions (S75), (S76) are met and give a physical interpretation for compressible Newtonian fluids: If

$$|k(\omega)|^2 \ll |\alpha(\omega)^2| = \frac{2|\omega|\rho}{|\tilde{g}_{\mathrm{s}}(\omega)|}, \qquad \text{(S78)}$$

$$|k(\omega)|^2 \gtrsim |\gamma(\omega)^2| = \frac{3|\omega|\rho}{|2\tilde{g}_{\mathrm{s}}(\omega) + \tilde{g}_{\mathrm{d}}(\omega)|}, \qquad \text{(S79)}$$

then $\lambda_t \approx \alpha$ and $|\lambda_l|$ is of the order of $|k|$. It is then clear that (S75) holds, while the dominant term in the bracket on the left hand side of eq. (S76) can be seen to be $\lambda_t^4 \approx \alpha^4$, so that this condition is also fulfilled. From eq. (S78) it furthermore follows that the left inequality in eq. (S73) implies the right one.

In the context of a Newtonian viscous fluid, where $\tilde{g}_{\mathrm{s}}$ is given by eq. (S25), eq. (S78) is equivalent to

$$\frac{\rho|\omega|}{\eta|k|^2} \gg 1, \qquad \text{(S80)}$$

which is the high Reynolds number condition, with typical velocity $|\omega|/|k|$ and typical length scale $|k|^{-1}$. Since $k$ is in general complex and the phase velocity of a surface wave is not the local velocity of the motion of the bulk medium, this interpretation can of course not be taken too literally. As for the interpreation of eq. (S79): Assuming the dominant term in the denominator on the right hand side of eq. (S79) to be the imaginary part of $\tilde{g}_{\rm d}$, i.e.

$$2\tilde{g}_{\rm s} + \tilde{g}_{\rm d} \approx \mathrm{Im}\,(\tilde{g}_{\rm d})\,, \tag{S81}$$

which for $\tilde{g}_{\rm s}$, $\tilde{g}_{\rm d}$ given by eqs. (S25-S27), with the parameters for water (S28-S31), is appropriate for $\omega \ll 10^{12}\ \mathrm{s}^{-1}$, eq. (S79) can be rewritten as

$$\frac{|\omega|^2}{|k|^2} \lesssim c_{bulk}^2. \tag{S82}$$

Since the phase velocity is

$$c(\omega) = \frac{\omega}{\mathrm{Re}(k(\omega))} \geq \frac{\omega}{|k(\omega)|}, \tag{S83}$$

we see that a sufficient condition for ineq. (S79) to hold is that the phase velocity of the wave solution should be smaller than the bulk sound velocity. In the limit of incompressibility, $c_{bulk} \to \infty$, we have $|\gamma| \to 0$, so that eq. (S79) is always fulfilled and $\lambda_l = k$. On the other hand, we see that if the phase velocity of a surface wave solution is much less than the sound velocity of the bulk medium, then $|k|^2 \gg |\gamma|^2$, so that $\lambda_l \approx k$ and compressibility effects of the bulk medium can be neglected in the study of surface waves.

The factorization (S77) yields two independent eqs. which lead to wave solutions, obtained by setting the first or second factor equal to zero. The first one yields a generalization of the capillary-gravity-flexural surface waves, the second a generalization of Lucassen's longitudinal capillary waves. We remark that $\lambda_l$ and the bending properties $\kappa_{2D}, \eta_{\rm 2D}^{\perp}$ (which are contained in $\tilde{\Pi}_{\rm 2D}$, eq. (S62)) only enter the former equation, while $\lambda_t$ and the in-plane viscoelastic response of the surface, $\tilde{g}_{\rm 2D}$, only enter the latter. As can be seen from this factorization, the Rayleigh solutions do not appear in this limit of a theory that includes bulk viscoelasticity, gravity, surface viscoelasticity and surface pressure. This factorization is a generalization of previous factorizations [16, 17]. That the factorization does not always hold was already predicted by Lucassen and is an established experimental result [18, 19].

*Capillary-gravity-viscosity (CGV) surface wave.* A sufficient condition to obtain the factorized conditional eq. (S77) was given by the inequalities (S73), (S78), (S79). In view of the definition of $\lambda_t$, eq. (S56), the second of these inequalities is equivalent to

$$\lambda_t = \sqrt{k^2 + \alpha^2} \approx \alpha. \tag{S84}$$

We now explore what happens if we assume the other term in the square root to be dominant, i.e. if we assume that

$$|k| \gg |\alpha|, \tag{S85}$$

so that

$$\lambda_t = \sqrt{k^2\left(1 + \left(\frac{\alpha}{k}\right)^2\right)} = \pm k\sqrt{1 + \left(\frac{\alpha}{k}\right)^2} \tag{S86}$$

$$\approx \pm k\left(1 - \frac{\alpha^2}{2k^2}\right), \tag{S87}$$

where the sign depends on the complex phases of the two complex numbers $k^2$, $1+(\alpha/k)^2$ and is due to our definition of the square root as the complex root with positive real part, cf. the discussion after eq. (S58). Since we found numerically that the minus sign is the correct choice for the CGV wave on water, we use it in the following.

To present a minimal model where the CGV wave appears, we assume the only non-negligible surface property to be the surface pressure $\sigma_{\rm 2D}$, i.e. we set $\rho_{\rm 2D}$, $\tilde{g}_{\rm 2D}$, $\kappa_{2D}$ equal to zero. Also, we limit ourselves here to the incompressible case, where $\lambda_l = k$. Note that since the real part of $\lambda_l$ should be positive for the wave to decay away from the interface, $\lambda_l = k$ means that we assume the real part of $k$ to be positive, i.e. we consider waves traveling into the positive $x_1$-direction. With these assumptions, and using the approximation (S87), the conditional eq. (S63) simplifies to a fourth order polynomial in $k$,

$$0 = 2\tilde{g}_{\rm s}^2 k^4 + \rho\sigma_{\rm 2D}k^3 - 3i\omega\rho\tilde{g}_{\rm s}k^2 + \rho^2 gk - \omega^2\rho^2. \tag{S88}$$

Since there exists a general formula for the roots of fourth order polynomials, the four complex roots $k(\omega)$ could be written down explicitly. However, we content ourselves here with deriving a simple approximate solution to eq. (S88) for the physically relevant special case of a water-like viscoelastic material.

For this, we use the water density given by eq. (S30), the shear relaxation function of water introduced in eq. (S25), with viscosity given by eq. (S28). We furthermore assume standard gravitational acceleration, $g = 9.81\,\mathrm{m/s^2}$, and surface tension, $\sigma_{\rm 2D} = 72$ mN/m. The other surface parameters, namely $\rho_{\rm 2D}$, $\eta_{\rm 2D}$, $\eta_{\rm 2D}'$, $\eta_{\rm 2D}^{\perp}$, $K_{\rm 2D}$, $\kappa_{2D}$, we set to zero.

Based on these parameter values and the numerically found solution $k(\omega)$ shown in fig. 3 (b), (e), we will now derive an approximate analytical solution for eq. (S88). Although the numerical solution for the CGV wave in fig. 3 (b), (e) was calculated using a compressible water-like viscoelastic medium as described by eqs. (S26), (S27), assuming incompressibility in the following derivation is appropriate, as the compressibility effects at the frequencies considered ($\omega \lesssim 1$ Hz) were numerically found to be negligibly small, c.f. the discussion following eq. (S81).

To begin our derivation, we denote by $p$, $q$ the real- and imaginary parts of $k$, i.e.

$$k(\omega) = p(\omega) + iq(\omega), \tag{S89}$$

with $p(\omega)$, $q(\omega)$ real and positive. From figs. 3 (b), (e) we then observe that the viscous surface wave solution should fulfill

$$|p(\omega)| \ll |q(\omega)|. \tag{S90}$$

Suppressing the frequency dependence in the notation from now on, we approximate the powers of $k$ by keeping only terms up to linear order in $p$,

$$k = p + iq, \tag{S91}$$
$$k^2 = \left[p^2 - q^2\right] + i\left[2pq\right] \approx -q^2 + i2pq, \tag{S92}$$
$$k^3 = \left[p(p^2 - 3q^2)\right] + i\left[q(-q^2 + 3p^2)\right] \approx -3pq^2 - iq^3, \tag{S93}$$
$$k^4 = \left[\left(p^2 - q^2\right)^2 - 4p^2q^2\right] + i\left[4pq\left(p^2 - q^2\right)\right] \approx q^4 - i4pq^3. \tag{S94}$$

Next, we insert these powers into the polynomial eq. (S88) and, using that $\sigma_{\mathrm{2D}}$, $\rho$, $g$, $\tilde{g}_{\mathrm{s}} = 2\eta$ are all real numbers, decompose the polynomial into eqs. for real- and imaginary part, yielding

$$0 = \left(8\eta^2 q^4 - \omega^2\rho^2\right) + p\left(-3\sigma_{\mathrm{2D}}\rho q^2 + 12\omega\eta\rho q + \rho^2 g\right), \tag{S95}$$
$$0 = (-32\eta^2 p - \rho\sigma_{\mathrm{2D}})q^3 + 6\omega\rho\eta q^2 + \rho^2 gq. \tag{S96}$$

In both equations, we make further simplifications by estimating the various sizes of the terms relative to each other based on the parameters for water and the expected sizes for $p$, $q$. We start with eq. (S96), which represents the imaginary part of the polynomial eq. (S88):

- Since
$$\eta^2 \approx 10^{-6}\ \mathrm{Pa}^2 \cdot \mathrm{s}^2, \tag{S97}$$
while
$$\rho\sigma_{\mathrm{2D}} \approx 10^1\ \mathrm{Pa}^2 \cdot \mathrm{s}^2/\mathrm{m}, \tag{S98}$$
and in view of the numerical solution, which indicates that $|p| \ll 10^6\ \mathrm{m}^{-1}$, we have
$$\eta^2 p/(\rho\sigma_{\mathrm{2D}}) \ll 1, \tag{S99}$$
so that we approximate the bracket in the first term as
$$-32\eta^2 p - \rho\sigma_{\mathrm{2D}} \approx -\rho\sigma_{\mathrm{2D}}. \tag{S100}$$
- The second term of eq. (S96) is of the order
$$6\omega\rho\eta q^2 \approx 6\omega q^2\ \mathrm{kg}^2\mathrm{m}^{-4}\mathrm{s}^{-1}, \tag{S101}$$
while the third term is of the order
$$\rho^2 gq \approx q10^7\ \mathrm{kg}^2\mathrm{m}^{-5}\mathrm{s}^{-2}. \tag{S102}$$
In view of the numerical solution we expect $q \approx 10^3\ \mathrm{m}^{-1}$, $\omega \lesssim 10\ \mathrm{s}^{-1}$, so that
$$(6\omega\rho\eta q^2)/(\rho^2 gq) \ll 1, \tag{S103}$$
and the second term can be neglected compared to the third term in eq. (S96).

Motivated by these arguments, we simplify the bracket and neglect the second term in eq. (S96). Then, the equation does not depend on $p$ anymore and can be solved directly for the imaginary part of $k(\omega)$, to give

$$q(\omega) = \sqrt{\frac{\rho g}{\sigma_{\mathrm{2D}}}}. \tag{S104}$$

This result for $q$ can now be inserted into eq. (S95), which then can be solved for the real part of $k(\omega)$ immediately, to yield

$$p(\omega) = \frac{4\eta^2 g}{\sigma_{\mathrm{2D}}^2} - \frac{\omega^2}{2g}, \tag{S105}$$

where we have used the approximation

$$2g - 12\omega\eta\sqrt{g/(\rho\sigma_{\mathrm{2D}})} \approx 2g, \tag{S106}$$

which is appropriate for $\omega \lesssim 10\ \mathrm{s}^{-1}$, since $\eta\sqrt{g/(\rho\sigma_{\mathrm{2D}})} \approx 10^{-2}\ \mathrm{m/s}$, so that

$$\omega\eta/\sqrt{g\rho\sigma_{\mathrm{2D}}} \ll 1. \tag{S107}$$

Combining eqs. (S104), (S105), we thus have derived the approximate dispersion relation

$$k(\omega) = \left[\frac{4\eta^2 g}{\sigma_{\mathrm{2D}}^2} - \frac{\omega^2}{2g}\right] + i\sqrt{\frac{\rho g}{\sigma_{\mathrm{2D}}}}. \tag{S108}$$

This is eq. (15) from the main text, and as can be seen in figs. 3 (b), (e) of the main text and figs. S1 (b), (e), it describes the behavior of the numerical solution quite accurately.

The solution (S108) can be inserted back into the approximations (S85), (S90), (S99), (S103), (S107), to see for which system parameters it is self-consistent. After short calculation it can then be concluded that if

$$\frac{g\eta^4}{\rho\sigma_{\mathrm{2D}}^3} \ll 1, \tag{S109}$$
$$\frac{\omega\sigma_{\mathrm{2D}}}{\eta g} \ll 1, \tag{S110}$$

then the result (S108) is consistent with the assumptions made in deriving it. In that sense eq. (S108) can, for fixed $g$, $\rho$, $\sigma_{\mathrm{2D}}$, be interpreted as a low $\eta$, low $\omega/\eta$ asymptotic solution to the dispersion relation eq. (S63).

**Existence regions for diminished surface tension.** – To facilitate comparison between different parameter regimes, the value $\sigma_{\mathrm{2D}} = 72$ mN/m for the surface tension was used for all numerical calculations presented in the main text. While this value is appropriate for pure water [5], in the presence of a monolayer the value of $\sigma_{\mathrm{2D}}$ is typically smaller, but of the same order of magnitude as the pure water value [20–22]. To show that a smaller value of $\sigma_{\mathrm{2D}}$ leads to qualitatively similar results as the ones presented in the main text, we recalculated figs. 3, 4 from the main text for the surface tension $\sigma_{\mathrm{2D}} = 10$ mN/m, and with otherwise identical parameters (namely $g = 9.81$ m/s$^2$, $\rho_{2D}$, $\eta_{2D}$, $\eta'_{2D}$, $\eta^{\perp}_{2D}$, $\kappa_{2D}$ equal to zero, water as described by eqs. (S25-S31)): Figure S1, which is the low-$\sigma_{2D}$ analogue of fig. 3, shows phase velocities and propagation distances for different surface configurations, while fig. S2, the low-$\sigma_{2D}$ analogue of fig. 4, shows the existence state diagram for the different wave solutions in the $(\omega, K_{2D})$-plane.

Comparing figs. 3 and S1, we see that the dispersion relations for the two values of $\sigma_{2D}$ are qualitatively similar. More specifically, subplots (b), (e) of fig. S1 show that for vanishing area elastic modulus, $K_{2D} = 0$, capillary-gravity and CGV wave exist, and that the wave number $k(\omega)$ of the CGV wave is very well approximated by eq. (S108). Comparing figs. 3, S1 (b), (e), we furthermore observe that the maximal frequency up to which the CGV wave exists, which we will denote by $\omega^{\mathrm{CGV}}_{\mathrm{max}}$ in the following, is larger for the smaller value of $\sigma_{2D}$. In view of our analytical dispersion relation, eq. (S108), this is expected: From figs. 3 (b), S1 (b), we see that the phase velocity of the CGV wave diverges as $\omega$ approaches $\omega^{\mathrm{CGV}}_{\mathrm{max}}$ from below. According to the definition of the phase velocity, eq. (S64), a divergence in phase velocity means that the real part of $k(\omega)$ is equal to zero. Equating the real part of $k(\omega)$, as defined by eq. (S108), with zero, and solving for $\omega$, we obtain

$$\omega^{\mathrm{CGV}}_{\mathrm{max}} = \sqrt{2}\frac{2\eta g}{\sigma_{2D}}. \tag{S111}$$

From this equation we see that $\omega^{\mathrm{CGV}}_{\mathrm{max}} \propto 1/\sigma_{2D}$, and using the parameters $\eta = 1$ mPa $\cdot$ s, $g = 9.81$ m/s$^2$, we get for $\sigma_{2D} = 10$ mN/m that $\omega^{\mathrm{CGV}}_{\mathrm{max}} \approx 2.8$ s$^{-1}$, while for $\sigma_{2D} = 72$ mN/m we obtain $\omega^{\mathrm{CGV}}_{\mathrm{max}} \approx 0.39$ s$^{-1}$. Both values are in good agreement with the numerical results shown in figs. 3, S1 (b) (e). Since the frequency range accessible in an experiment is usually limited, the fact that a lower surface tension leads to higher $\omega^{\mathrm{CGV}}_{\mathrm{max}}$ may be useful when designing an experiment to measure the CGV wave.

The inset of fig. S2 (a) shows that also for $\sigma_{2D} = 10$ mN/m, there is a parameter range where three solutions $k(\omega)$ coexist. Comparing figs. 4, S2 (a), it can be inferred that for the smaller value of $\sigma_{2D}$, this coexistence region is shifted towards higher angular frequencies $\omega$ and higher values of the area elastic modulus $K_{2D}$.

**Crossover from capillary-gravity and Lucassen wave to Rayleigh wave.** – As can be seen in fig. 3 (a) of the main text and fig. S2 (a), the Lucassen wave transforms into the Rayleigh wave at high frequencies. In fig. S3, we illustrate this for $\sigma_{2D} = 72$ mN/m, $K_{2D} = 10$ mN/m. The figure shows the crossover of both capillary-gravity and Lucassen waves into the two Rayleigh waves that exist on a water-like viscoelastic half-space (i.e. the Rayleigh waves shown in figs. 2 (a), (d) of the main text). It can be seen that the capillary-gravity wave crosses over to become the $n = 3$ Rayleigh wave at $\omega \approx 10^{10}$ s$^{-1}$, while the Lucassen wave transforms into the $n = 2$ Rayleigh wave at $\omega \approx 10^{9}$ s$^{-1}$.

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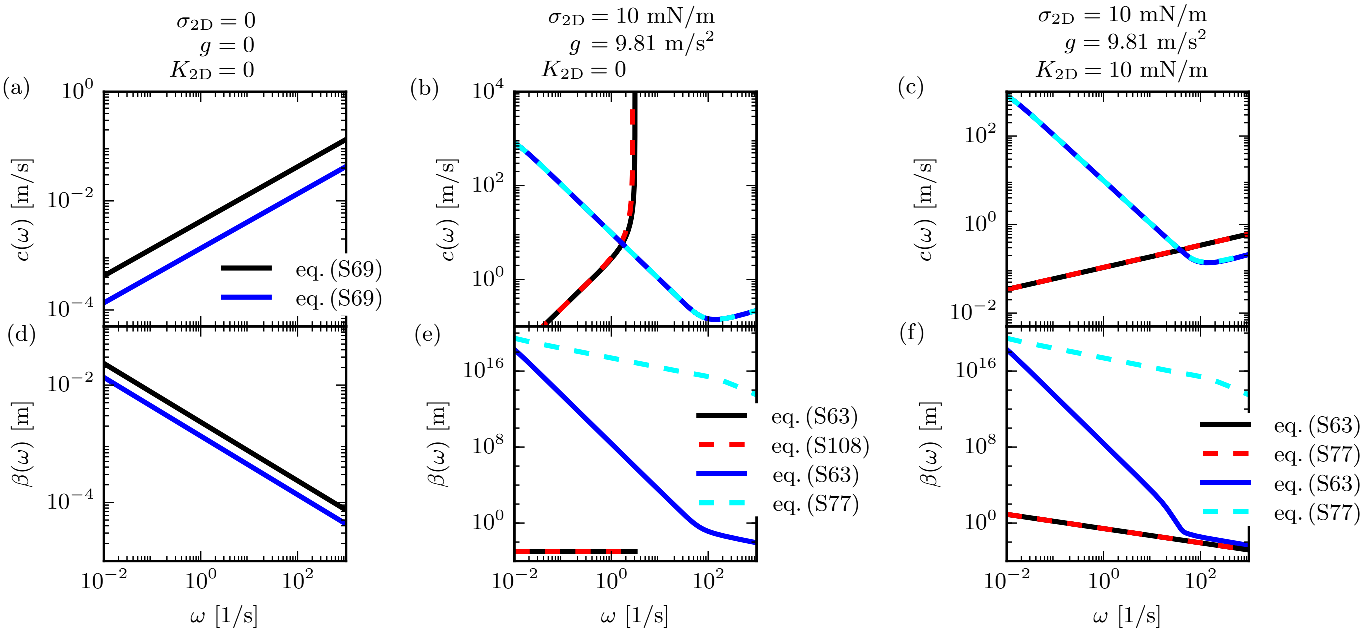


Fig. S1: **Phase velocities and propagation distances on a water-like viscoelastic half-space for different surface parameters.** (a), (d) show the properties of the two Rayleigh waves that exist on a water-like viscoelastic half-space. (b), (e) show the capillary-gravity wave and the viscous surface wave on a water-like viscoelastic half-space with surface tension $\sigma_{2D} = 10$ mN/m and gravitational acceleration $g = 9.81$ m/s$^2$. (c), (f) show the capillary-gravity wave and the Lucassen wave on a water-like viscoelastic half-space with an interface (surface tension $\sigma_{2D} = 10$ mN/m, area elastic modulus $K_{2D} = 10$ mN/m) and gravitational acceleration $g = 9.81$ m/s$^2$. The curves in subplots (a), (d) were calculated using the analytical solution of eq. (S66). The analytic formula for the CGV wave, eq. (S108), is shown as dashed red lines in subplots (b), (e). All other curves are obtained from numerically solving either the full dispersion relation, eq. (S63), or one of the two factors from the factorization, eq. (S77), with the interfacial parameters $\rho_{2D}$, $\eta_{2D}$, $\eta'_{2D}$, $\eta^{\perp}_{2D}$, $\kappa_{2D}$ set to zero. In all cases, phase velocities and propagation distances are calculated from $k(\omega)$ via eqs. (S64), (S65).

Fig. S2: **Existence state diagram for surface waves.** (a) shows an existence state diagram for surface waves for fixed $\sigma_{2D} = 10$ mN/m, $g = 9.81$ m/s$^2$, as a function of interface elastic modulus $K_{2D}$ and frequency $\omega$. The capillary-gravity wave exists throughout the plane, its existence region is therefore not plotted. The hatched regions indicate where both the phase velocities and propagation distances of the numerical solutions of eq. (S63) have less than 30% relative deviation from the phase velocities and propagation distances of the numerical solutions of the limiting eqs. (S66), (S77), or the analytical formula (S108), respectively. Upper and lower red dashed lines indicate the positions of the dispersion relations from figs. S1 (b), (e) (lower red line) and (c), (f) (upper red line) in the phase diagram. The inset shows that there is a region where three waves coexist. Subplots (b), (d) illustrate this by showing the phase velocities and propagation distances of the three waves as a function of $\omega$ for $K_{2D} = 6 \times 10^{-4}$ mN/m, with this value for $K_{2D}$ indicated by the short red dashed line in subplot (a). The shaded regions and the curves in subplots (b), (d) are obtained by numerically solving eq. (S63). For all calculations, a water-like viscoelastic half-space is used as bulk medium and $\rho_{2D}$, $\eta_{2D}$, $\eta'_{2D}$, $\eta^{\perp}_{2D}$, $\kappa_{2D}$ are set to zero. Phase velocities and propagation distances are calculated from $k(\omega)$ using eqs. (S64), (S65).

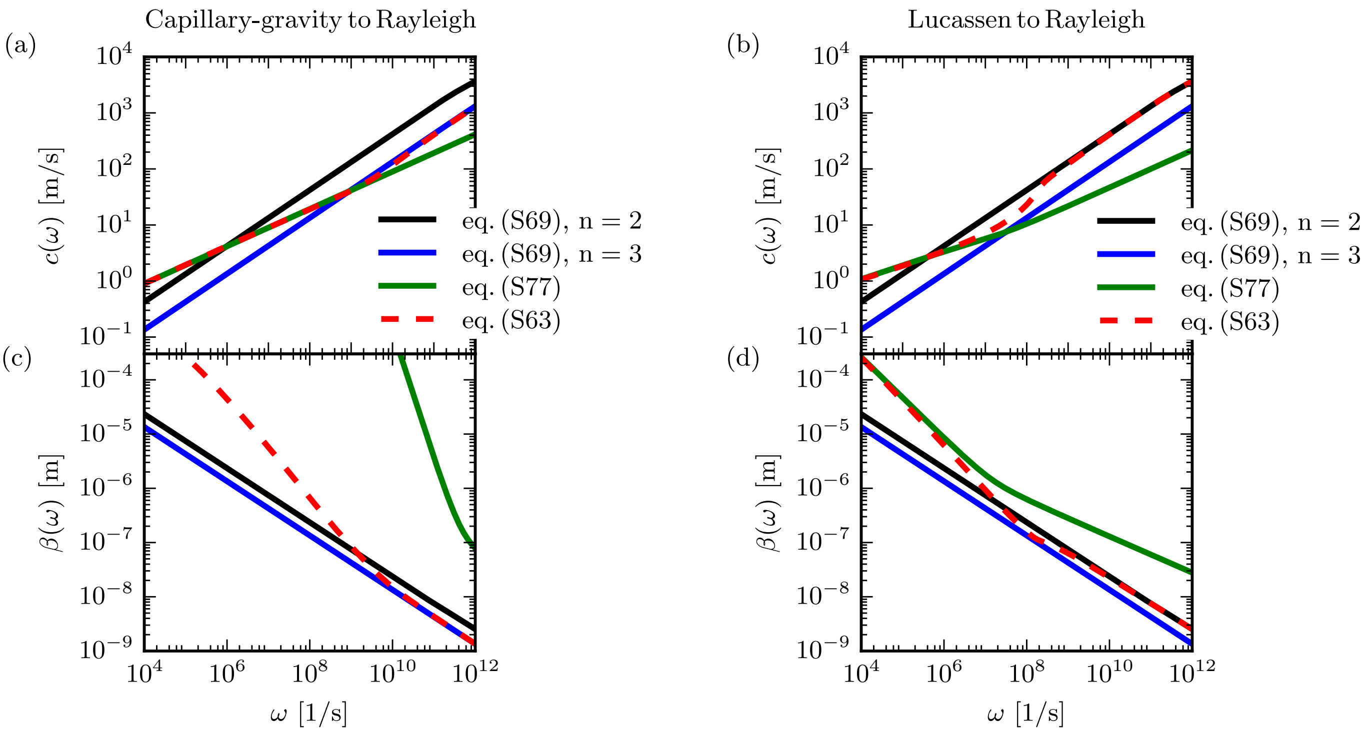


Fig. S3: **Frequency dependent crossover of capillary-gravity and Lucassen wave to Rayleigh wave on a water-like viscoelastic half-space.** (a), (c) show the phase velocities and decay lengths of the Rayleigh waves (black and blue lines), together with the factorized capillary-gravity wave dispersion relation (green line) and the corresponding solution of the exact dispersion relation (red dashed line). Note that the factorized capillary-gravity wave dispersion relation predicts a much larger decay length and is therefore only visible for frequencies $\omega \gtrsim 10^{10}$ s$^{-1}$ in subplot (c), c.f. fig. 2 (e). Subplots (b), (d) show also the phase velocities and decay lengths of the Rayleigh wave (black and blue lines), but together with the factorized Lucassen wave dispersion relation (green line) and the corresponding solution of the exact dispersion relation (red dashed line). For the Rayleigh waves, eq. (S69) is used. For the other curves, eqs. (S63), (S77) are solved numerically using the parameters $\sigma_{2\mathrm{D}} = 72$ mN/m, $g = 9.81$ m/s$^2$, $K_{2\mathrm{D}} = 10$ mN/m, with all other surface parameters set to zero. For all curves, the bulk medium is a water-like viscoelastic material as described by eqs. (S25), (S26) with parameters given by eqs. (S28-S31). Phase velocities and decay lengths are calculated from $k(\omega)$ according to eqs. (S64), (S65).

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