Blatt 8

1

a) $\hat{e}_x \times \hat{e}_y = \begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} = \hat{e}_z$

b) $\hat{e}_y \times \hat{e}_x = \begin{pmatrix}0\\1\\0\end{pmatrix} \times \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\-1\end{pmatrix} = -\hat{e}_z$

c) $\hat{e}_z \times \hat{e}_x = \begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} = \hat{e}_y$

d) $\hat{e}_z \times \hat{e}_y = \begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\0\end{pmatrix} = -\hat{e}_x$

e) $\hat{e}_z \times \hat{e}_z = \begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} = 0$

f) $\hat{e}_z \times (\hat{e}_y + \hat{e}_x) = \begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix} = -\hat{e}_x + \hat{e}_y$!

g) $(\hat{e}_x - \hat{e}_y) \times (\hat{e}_z + \hat{e}_x) = \begin{pmatrix}1\\-1\\0\end{pmatrix} \times \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\\1\end{pmatrix} = -\hat{e}_x - \hat{e}_y + \hat{e}_z$

2

$\vec{p} = \begin{pmatrix}2\\1\\0\end{pmatrix} \qquad q = \begin{pmatrix}0\\-1\\3\end{pmatrix}$

a) $\vec{p} \times \vec{q} = \begin{pmatrix}2\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\-1\\3\end{pmatrix} = \begin{pmatrix}1\cdot 3 - 0\cdot(-1)\\ 0\cdot 0 - 2\cdot 3\\ 2(-1) - 1\cdot 0\end{pmatrix} = \begin{pmatrix}3\\-6\\-2\end{pmatrix}$

$\vec{p} \cdot (\vec{p} \times \vec{q}) = \begin{pmatrix}2\\1\\0\end{pmatrix} \cdot \begin{pmatrix}3\\-6\\-2\end{pmatrix} = 6 - 6 = 0$

b) $A = |\vec{p} \times \vec{q}| = \sqrt{\begin{pmatrix}3\\-6\\-2\end{pmatrix} \cdot \begin{pmatrix}3\\-6\\-2\end{pmatrix}} = \sqrt{9+36+4} = \sqrt{49} = 7$

(3)

a) $|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\,\underbrace{\sin\frac{\pi}{6}}_{\text{Blatt 1}} = 2\cdot 3\cdot 1/2 = 3$

b) Gegenbeispiel: $\vec{a} = \hat{e}_x \quad \vec{b} = \vec{c} = \hat{e}_y$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{pmatrix}1\\0\\0\end{pmatrix}\times\left[\begin{pmatrix}0\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix}\right] = 0$$

aber $(\vec{a}\times\vec{b})\times\vec{c} = \left[\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix}\right]\times\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\0\end{pmatrix} = -\hat{e}_x$

(4)

a) Spatprodukt gibt Volumen von parallelepiped („Spat"):

$\vec{p} = \begin{pmatrix}5\\0\\0\end{pmatrix}, \ \vec{q} = \begin{pmatrix}3\\2\\0\end{pmatrix}$

$\vec{r} = \begin{pmatrix}1\\2\\2\end{pmatrix}$

$$V = |\vec{p}\cdot(\vec{q}\times\vec{r})|$$

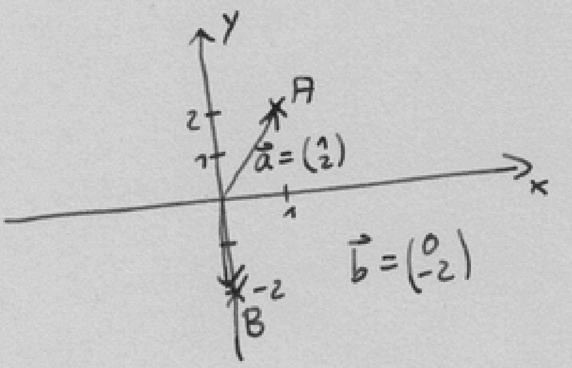
$$= \left|\begin{pmatrix}5\\0\\0\end{pmatrix}\cdot\left(\begin{pmatrix}3\\2\\0\end{pmatrix}\times\begin{pmatrix}1\\2\\2\end{pmatrix}\right)\right|$$

$$= \left|\begin{pmatrix}5\\0\\0\end{pmatrix}\cdot\begin{pmatrix}4\\-6\\4\end{pmatrix}\right| = |5\cdot 4| = 20$$

b) wie a): $V = \left|\begin{pmatrix}-1\\1\\0\end{pmatrix}\cdot\left[\begin{pmatrix}1\\-2\\0\end{pmatrix}\times\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}\right]\right| = \left|\begin{pmatrix}-1\\1\\0\end{pmatrix}\cdot\left[\begin{pmatrix}1\\-2\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}\right]\right|$

$$= \left|\begin{pmatrix}-1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-2\\-1\\0\end{pmatrix}\right| = |2-1| = 1$$

(5)



Gerade durch A & B:

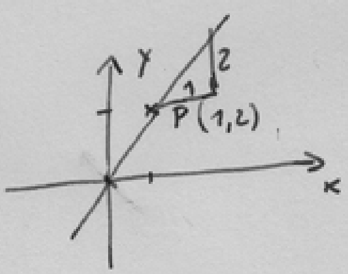
Vektordarstellung: $g = \{\vec{b} + t(\vec{b} - \vec{a})$ für $t \in (-\infty, \infty)\}$

Komponentendarstellung: $y = -2 + 4x$

y-Achsenabschnitt

Steigung: $4 = \frac{\Delta y}{\Delta x} = \frac{2-(-2)}{1}$

(6)



(a) Funktionale Form:

$g(x) = mx + b$

$m = 2$ (Steigung)

Punkt (1,2) : $g(1) = 2 = 2 \cdot 1 + b$

$\Rightarrow b = 0$

$\curvearrowright g(x) = 2x + 0 = 2x$ (Die Gerade geht durch den Ursprung)

Parameterdarstellung:

$g = \{\binom{1}{2} + t\binom{1}{2}$ für $t \in (-\infty, \infty)\}$

(b) $\hat{n} = \binom{n_1}{n_2}$ erfüllt: i) $n_1^2 + n_2^2 = 1$

ii) $\hat{n} \cdot \binom{1}{2} = 0$

$\Leftrightarrow n_1 + 2n_2 = 0$

(ii) in (i): $n_1^2 + n_1^2/4 = 1$

$\Rightarrow n_1^2 = \frac{4}{5} \Rightarrow n_1 = \pm \frac{2}{\sqrt{5}}$

in (ii) $\Rightarrow n_2 = \mp \frac{1}{\sqrt{5}}$

$\Rightarrow \hat{n} = \pm \binom{2/\sqrt{5}}{-1/\sqrt{5}}$ Es gibt zwei normierte Normalenvektoren.

7

a) Um eine Ebene aufzuspannen, müssen zwei Vektoren linear unabhängig sein, d.h. $\vec{a}_1 \times \vec{a}_2 \neq \vec{0}$

$$\vec{a}_1 \times \vec{a}_2 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3-8 \\ 4-1 \\ 4-6 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ -2 \end{pmatrix}$$ ⇝ Ebene ✓

$$\vec{a}_1 \times \vec{a}_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \times \begin{pmatrix} -2 \\ -6 \\ -4 \end{pmatrix} = \begin{pmatrix} -12+12 \\ -4+4 \\ -6-(-6) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ ⇝ spannen keine Ebene auf

$$\vec{a}_2 \times \vec{a}_3 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} -2 \\ -6 \\ -4 \end{pmatrix} = \begin{pmatrix} -16+6 \\ -2+8 \\ -12+8 \end{pmatrix} = \begin{pmatrix} -10 \\ 6 \\ -4 \end{pmatrix}$$ ⇝ Ebene ✓

b) $$\hat{n}_{12} = \frac{\vec{a}_1 \times \vec{a}_2}{|\vec{a}_1 \times \vec{a}_2|} = \frac{\begin{pmatrix} -5 \\ 3 \\ -2 \end{pmatrix}}{\sqrt{25+9+4}} = \begin{pmatrix} -5/\sqrt{38} \\ 3/\sqrt{38} \\ -2/\sqrt{38} \end{pmatrix}$$

$$\hat{n}_{23} = \frac{\vec{a}_2 \times \vec{a}_3}{|\vec{a}_2 \times \vec{a}_3|} = \frac{\begin{pmatrix} -10 \\ 6 \\ -4 \end{pmatrix}}{\sqrt{100+36+16}} = \frac{\begin{pmatrix} -10 \\ 6 \\ -4 \end{pmatrix}}{\sqrt{4 \cdot 38}} = \frac{\begin{pmatrix} -10 \\ 6 \\ -4 \end{pmatrix}}{2\sqrt{38}} = \begin{pmatrix} -5/\sqrt{38} \\ 3/\sqrt{38} \\ -2/\sqrt{38} \end{pmatrix}$$

8

7.5

a) $\begin{pmatrix} 8 & -3 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 34 \end{pmatrix}$

$\cdot 3/2$

$$\left.\begin{array}{cc|c} 8 & -3 & 11 \\ \frac{15}{2} & 3 & 51 \end{array}\right.$$

I+II:

$$\left.\begin{array}{cc|c} 8 & -3 & 11 \\ \frac{31}{2} & 0 & 62 \end{array}\right. \quad \hat{=} \quad \begin{array}{l} 8x - 3y = 11 \\ \frac{31}{2}x = 62 \end{array}$$

$$\Leftrightarrow \begin{array}{l} 8x - 3y = 11 \\ x = 4 \end{array} \quad \Leftrightarrow \begin{array}{l} 32 - 3y = 11 \\ x = 4 \end{array}$$

$$\Leftrightarrow \begin{array}{l} x = 4 \\ y = 7 \end{array}$$

b)

$$\begin{array}{c} (\mathrm{I}) \\ (\mathrm{II}) \\ (\mathrm{III}) \end{array} \left(\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 1 & -1 & 1 & 0 \\ -3 & -5 & 2 & 1 \end{array}\right)$$

$2\mathrm{II} - \mathrm{I}$

$\mathrm{III} + 3 \cdot \mathrm{II}$

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & -8 & 5 & 1 \end{array}\right) \begin{array}{c} (\mathrm{I}') \\ (\mathrm{II}') \\ (\mathrm{III}') \end{array}$$

$\mathrm{III}' - \frac{8}{5}\mathrm{II}'$

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & 0 & 5-\frac{24}{5} & 1+\frac{16}{5} \end{array}\right)$$

$$\Leftrightarrow \quad \begin{aligned} 2x + 3y - z &= 2 & (\mathrm{I}'') \\ -5y + 3z &= -2 & (\mathrm{II}'') \\ \tfrac{1}{5} z &= \tfrac{21}{5} & (\mathrm{III}'') \end{aligned} \Rightarrow \boxed{z = 21}$$

in II'': $-5y + 63 = -2 \Rightarrow \boxed{y = 13}$

in I'': $2x + 39 - 21 = 2 \Rightarrow 2x + 18 = 2$

$$\boxed{x = -8}$$

(9)

Winkelsumme im Dreieck:

$$\alpha+\beta+\gamma = 180^\circ \quad (I)$$

$$\alpha-2\beta+0\gamma = 0 \quad (II) \longleftarrow \text{„}\alpha \text{ doppelt so groß wie } \beta\text{"} \Leftrightarrow \alpha=2\beta$$

$$\alpha+\beta-\gamma = 0 \quad (III) \longleftarrow \alpha+\beta=\gamma$$

„zusammen sind α und β so gross wie γ"

$$\alpha \quad +\beta+\gamma = 180^\circ \quad (I')$$

$$0 \quad -3\beta-\gamma = -180^\circ \quad (II') \quad II-I$$

$$0 \quad 0 \quad -2\gamma = -180^\circ \quad (III') \quad III-I$$

$III' \Rightarrow \boxed{\gamma = 90^\circ}$; in II': $-3\beta - 90^\circ = -180^\circ$

$\Leftrightarrow \boxed{30^\circ = \beta}$

in (I'): $\alpha + 30^\circ + 90^\circ = 180^\circ \Rightarrow \boxed{\alpha = 60^\circ}$

(10)

a)
$$2x - y = 8 \quad (I)$$
$$6x - 5y = 32 \quad (II)$$

$\longrightarrow$
$$2x - y = 8 \quad (I')$$
$II-3I$: $0 - 2y = 8 \quad (II') \Rightarrow \boxed{y=-4}$

in I': $\boxed{x=2}$

b)
$$4x - 3y = 5 \quad (I)$$
$$8x - y = 3 \quad (II)$$

$\longrightarrow$
$$4x - 3y = 5 \quad (I')$$
$II-2I$ $0 + 5y = -7 \quad (II') \Rightarrow \boxed{y=-\frac{7}{5}}$

in I': ~~[illegible]~~

$$4x + \frac{21}{5} = 5$$
$$4x = \frac{4}{5} \Rightarrow \boxed{x = 1/5}$$

(10) Fortsetzung

c)

$$2x + y - 3z = 0 \quad (I)$$
$$6x + 3y - 8z = 0 \quad (II)$$
$$2x - y + 5z = 8 \quad (III)$$

$$2x + y - 3z = 0 \quad (I')$$
II − 3I: $$0 + z = 0 \quad (II')$$
III − I: $$-2y + 8z = 8 \quad (III')$$

umsortieren: $$2x + y - 3z = 0 \quad (I'')$$
$$-2y + 8z = 8 \quad (II'')$$
$$z = 0 \quad (III'') \Rightarrow \boxed{z = 0}$$

in II'': $\boxed{y = -4}$, in I'': $2x - 4 = 0 \Rightarrow \boxed{x = 2}$

d)

$$x + 2y - z = 0 \quad (I)$$
$$3x + 7y + 2z = 2 \quad (II)$$
$$4x - 2y + z = 1 \quad (III)$$

$$x + 2y - z = 0 \quad (I')$$
II − 3I: $$y + 5z = 2 \quad (II')$$
III − 4I: $$-10y + 5z = 1 \quad (III')$$

$$x + 2y - z = 0 \quad (I'')$$
$$y + 5z = 2 \quad (II'')$$
III' + 10II': $$55z = 21 \quad (III'') \Rightarrow \boxed{z = \frac{21}{55}}$$

in II'': $y + \frac{21}{11} = 2$

$\Rightarrow \boxed{y = \frac{1}{11}}$

in I'': $x + \frac{2}{11} - \frac{21}{55} = 0$

$x - \frac{11}{55} = 0$

$\boxed{x = +\frac{1}{5}}$