Ground state energy of an interacting electron system in the background of two opposite magnetic strings

Jürgen Dietel

Institut für Theoretische Physik,
Freie Universität Berlin,
Arnimallee 14, D 14195 Berlin, Germany[*]

Motivated by our earlier work, we show in this paper rigorously that the ground state energy and degeneracy of an infinitely extended system of interacting electrons in the background of a homogeneous magnetic field and two separated magnetic strings of opposite strength is the same as for the system without strings. By using symmetry considerations we obtain further that the energy spectrum does not depend on the string separation distance for strictly positive distances. As a side effect of our considerations, we obtain a virial theorem for the two string system in the case of a homogeneous interaction potential which has the same form as the virial theorem without the strings.

I. INTRODUCTION

Since the discovery of the Aharonov-Bohm[1] effect there is a growing interest in quantum mechanical systems in the presence of a finite number of magnetic strings. Due to the localization of the magnetic field of a string these systems are rather quantum mechanical objects than classical ones. The magnetic strings affect the energy spectrum and the electronic wavefunctions due to the extension of the wavefunctions which is governed by the uncertainty principle. In this paper we consider an infinitely extended system of two dimensional interacting electrons under the influence of a homogeneous magnetic field B and two magnetic strings of opposite strength separated by a distance d. We will show that the spectrum of the electron system does not depend on d for $d > 0$. Furthermore, we will show that the ground state energy and the degeneracy of the electron system are the same with and without the two strings[5].

We have two motivations to consider this two string system. The first motivation comes from the observation[2,3] that the Green's function of Chern-Simons theories for fractional quantum Hall systems in the commonly used Coulomb gauge vanishes exponentially with an exponent proportional to the logarithm of the area of the system. We have analyzed in Ref. 4 the physically better motivated Chern-Simons Green's function in the temporal gauge. The Green's function is divergent to lowest order perturbation theory at zero temperature. In contrast to this we have shown in Ref. 4 non-perturbatively that in the case of the validity of the two string spectral properties mentioned above we obtain a finite Green's function in the temporal gauge. It was shown that for non-interacting electrons the ground state energy is the same with and without the two strings. Furthermore, we calculated by an explicit construction of the ground state wavefunctions the interaction free Chern-Simons Green's function in the temporal gauge. At last, we gave two non-rigorous physical arguments that the interacting electron system should also have the same ground state energy with and without the two strings. This will be shown rigorously in this paper.

The second motivation for this paper comes from the observation made in Ref. 6 that in the continuum limit of a tight binding model, the force of a screw dislocation on the electrons can be described as a magnetic string located at the core of the dislocation. This force is caused by the distorted topology of the real space[7,8]. In this approximation the crystal lattice is completely lost. This model was refined recently[9,10] by taking into account also the distance dependence of the transfer matrix of the tight binding model in the continuum limit. In the context of dislocations the interaction energy of two opposite screw dislocations is an interesting quantity. This energy was first calculated using elasticity theory[11]. Recently, it became possible to carry out ab initio calculations on parallel computers to calculate the interaction energy of two opposite dislocations for special crystals which take also the electron system into account (e.g.[12]). These calculations neglect the interaction potential between the electrons. Our analysis will thus also be useful in the field of dislocations in solids.

The strategy of the proof that the ground state energy is the same with and without the two strings is as follows: In section II we show by symmetry considerations that the energy spectrum of the two string system does not depend on the string separation for d strictly positive. This is done by calculating the derivate of the energy function with respect to the string separation d. By using the space inversion symmetry of the two string system we are able to express this derivate as a function which depends on the asymptotics of the wavefunction at the two strings. This will be shown in subsection A of section II. In subsection B we go further by rewriting this function as a differential of the energy with respect to the magnetic field and the interaction strength. Thus, we obtain a differential equation

for the energy function. By using the space inversion symmetry we obtain from this differential equation that the energy spectrum does not depend on the string separation d. In section III we show that the ground state energy is continuous at string separation zero. The technical details of the proof are deferred to appendix A. By using the results of section II and section III we obtain in section IV that the ground state energy and degeneracy is the same for the system with or without the two strings. We give a conclusion in section V.

II. THE SEPARATION DISTANCE INDEPENDENCE OF THE ENERGY SPECTRUM

In order to show the spectral properties mentioned above for the infinite system we consider in this paper a high symmetric finite system with zero boundary conditions. The boundary should have an inversion symmetry. The two strings are arranged in order to ensure that the inversion symmetry is indeed conserved. In this paper we show the spectral properties for this system. We loose the boundary dependence when carrying out the limit to an infinite system size.

The Hamiltonian of N electrons in the background of a homogeneous magnetic field $\vec{B} = -B\vec{e}_z$ $(B > 0)$ and two magnetic strings of opposite strength $\pm\phi$ separated by d is given by

$$H_{ss}(\phi, d) = \sum_i^N \frac{1}{2m}\left(-i\vec{\nabla}_i + \vec{A}(\vec{r}_i) + \phi\vec{f}(\vec{r}_i + d\vec{e}_x/2) - \phi\vec{f}(\vec{r}_i - d\vec{e}_x/2)\right)^2 + \frac{\alpha}{2}\sum_{i \neq j} V^{ee}(\vec{r}_i - \vec{r}_j)\,,\tag{1}$$

where the convention $\hbar = 1$, $c = 1$ and $e = 1$ has been used. $\vec{A}(\vec{r})$ is the vector potential $\vec{A} = \vec{B} \times \vec{r}/2$. The function $\vec{f}(\vec{r})$ represents the vector potential of a magnetic string with unit flux quantum and is given by $\vec{f}(\vec{r}) = \vec{e}_z \times \vec{r}/r^2$. $V^{ee}(\vec{r})$ is a potential function. In the following, we restrict us at first on a homogeneous potential functions of scaling dimension $\gamma \in \mathbb{R}$, i.e. $V^{ee}(a\vec{r}) = |a|^{-\gamma}V^{ee}(\vec{r})$. In the case of the Coloumb interaction we have $\gamma = 1$. α is the coupling constant of the interaction potential. In the following derivations we consider only systems with string strength $0 \leq \phi < 1$. By using the fact that the Hamiltonian (1) with string strength ϕ is unitary equivalent to the Hamiltonian with strength $\phi + z, z \in \mathbb{Z}$ we immediately obtain a generalization of all results deduced in this paper to arbitrary ϕ. The transformation between both systems is given by a phase transformation.

Next, we have to determine the domain of the Hamiltonian H_{ss} (1), which can be done in two ways. We restrict ourselves to a short outline due to lack of space. A much more detailed discussion can be found in the literature cited below.

First, one can determine the self-adjoint domains of H_{ss} by a mathematical extension formalism[13]. There are many different self-adjoint extensions of H_{ss}. Which of these extensions has to be used, depends on the particular physical system[14]. For example, an additional point force such as a Zeeman term at the origin of the string can change the extension that has to be used.

In this paper, we pursue a more physical way of determining the domain of (1). It consists of a regularization of the magnetic strings to a tube of finite width. This yields a Hamiltonian with finite regular potentials for which the self-adjoint domain of the Hamiltonian is well known. After determining the eigenvalues and eigenfunctions of this regularized Hamiltonian, one can carry out the limit of tube width zero. This was done first for the one string system by Aharonov and Bohm[1] and for much more general regularizations by Hagen in Ref. 15. Similar considerations for the two string system were outlined by the author[4] and can be extended easily to the many particle (interacting) case. We obtain that the physical domain $\mathcal{D}[H_{\phi,d}]$ of the Hamiltonian H_{ss} consists of wavefunctions with nodes at the positions of the two strings

$$\mathcal{D}[H_{\phi,d}] := \{\Psi(\vec{r}) \mid \Psi(\vec{r}) \text{ is square integrable and } \Psi(-d\vec{e}_x/2) = \Psi(d\vec{e}_x/2) = 0\}\,.\tag{2}$$

It is then easily seen by carrying out a partial integration that the Hamilton operator (1) is symmetric on this domain.

Next, we will determine the asymptotics of the eigenfunctions of H_{ss} at the position of the two strings. Without loss of generality we determine the asymptotics for $\vec{r}_1 \to \mp d\vec{e}_x/2$. We now seek a solution of the Schrödinger equation (1) of the asymptotic form

$$\Psi(r) = \frac{1}{\sqrt{2\pi}}f_p^{\pm}(|\vec{r}_1^{\pm}|; \vec{r}_2, .., \vec{r}_N)e^{ip\varphi_{\pm}}\tag{3}$$

where $\vec{r}_1^{\pm} := \vec{r}_1 \mp d\vec{e}_x/2$ using cylindrical polar coordinates. $f_p^{\pm}$ is the asymptotic wave function for $\vec{r}_1 \to \mp d\vec{e}_x/2$. Then one gets for the eigenvalue equation in polar coordinates at $\vec{r}_1 \approx \mp d\vec{e}_x/2$

$$-\frac{1}{2m}\left(\partial^2_{|\vec{r}_1^{\pm}|} + \frac{1}{|\vec{r}_1^{\pm}|}\partial_{|\vec{r}_1^{\pm}|} - \frac{(p \mp \phi)^2}{|\vec{r}_1^{\pm}|^2}\right)f_p^{\pm}(|\vec{r}_1^{\pm}|; \vec{r}_2, .., \vec{r}_N)$$

$$+ \sum_{p'} \left(M_{p,p'}(|\vec{r}_1^{\pm}|) f_{p'}(|\vec{r}_1^{\pm}|) + D_{p,p'}(|\vec{r}_1^{\pm}|) \partial_{|\vec{r}_1^{\pm}|} \right) f_{p'}^{\pm}(|\vec{r}_1^{\pm}|; \vec{r}_2, .., \vec{r}_N) = E\, f_p^{\pm}(|\vec{r}_1^{\pm}|; \vec{r}_2, .., \vec{r}_N). \tag{4}$$

$M_{p,p'}$ is a multiplication operator which scales as $O(1/|\vec{r}_1^{\pm}|)$ for $|\vec{r}_1^{\pm}| \to 0$ or a differential operator with respect to the coordinates $\vec{r}_2, .., \vec{r}_N$. Similarly $D_{p,p'}$ is a multiplication operator which scales as $O(|\vec{r}_1^{\pm}|^0)$. Both of these operators reduce the scaling power of the function $f_p^{\pm}$ at most by one. The asymptotic eigenvalue equation (3) and the restriction on the wavefunction (2) then immediately leads to

$$f_p^{\pm} = v_p^{\pm}(\vec{r}_2, .., \vec{r}_N) \left(\delta_{p,0}|\vec{r}_1^{\pm}|^{\phi} + \delta_{p,\pm 1}|\vec{r}_1^{\pm}|^{1-\phi} \right) + O(|\vec{r}_1^{\pm}|). \tag{5}$$

for $d > 0$. Here $v_p^{\pm}$ is a finite function of $\vec{r}_2, .., \vec{r}_N \neq \pm\vec{d}/2$. We see from (5) that the eigenfunctions have not to be continuous in general at $d = 0$ for $0 < \phi < 1$. The same holds also for the eigenvalues.

In the following, we denote by $\Psi_{\phi,d}^n$ the normalized eigenstates with energies $E^n(\phi,d)$ labelled by n. With the help of the current operator $\hat{\vec{J}}_{\phi,d}(\vec{r}) = \sum_i \delta(\vec{r} - \vec{r}_i)(-i\vec{\nabla}_i + \vec{A}(\vec{r}_i) + \phi\vec{f}(\vec{r}_i^-) - \phi\vec{f}(\vec{r}_i^+))/m$ (the physical current being the real part of the expectation value of this current operator) we get

$$\partial_{\xi} E^n = \int d^2 r (\partial_{\xi}\vec{F}_{\phi,d}(\vec{r})) \langle \hat{\vec{J}}(\vec{r}) \rangle_{\phi,d}^n + S_{\xi}^n \tag{6}$$

with

$$\vec{F}_{\phi,d}(\vec{r}) = \frac{\phi}{m} \left(\vec{f}(\vec{r}^-) - \vec{f}(\vec{r}^+) \right) \tag{7}$$

and

$$S_{\xi}^n(\phi) = \sum_{\sigma \in \{\pm\}} \sum_{i=1}^{N} \int d^2 r_1 .. d^2 r_{i-1} d^2 r_{i+1} .. d^2 r_N \left[\int_{S_{\sigma\vec{d}/2}} d\vec{o}_i \left((\vec{\nabla}_i \Psi_{\phi,d}^n)^* \partial_{\xi} \Psi_{\phi,d}^n - (\Psi_{\phi,d}^n)^* \vec{\nabla}_i \partial_{\xi} \Psi_{\phi,d}^n \right) \right] \tag{8}$$

where ξ equals d or ϕ, respectively. Here, we denoted $\langle \Psi_{\phi,d}^n | \hat{\vec{J}}_{\phi,d}(\vec{r}) | \Psi_{\phi,d}^n \rangle$ by $\langle \hat{\vec{J}}(\vec{r}) \rangle_{\phi,d}^n$. The integration contour of the surface integral $\int_{S_{\sigma\vec{d}/2}} d\vec{o}_i$ is an infinitesimally small circle around the string position $\vec{r}_i = \sigma d\vec{e}_x/2$. With the help of the asymptotics (5) of the eigenfunctions, we can show easily that S_{ϕ}^n is zero. This is not the case for S_d^n. The reason is that the domain (2) of the Hamiltonian (1) changes for different string positions, i.e. the derivation of the eigenfunction $\Psi_{\phi,d}^n$ with respect to d can be divergent at the string position and thus is not in the domain of the Hamiltonian (1). This results in surface terms S_d^n. With the help of the definition of the restricted overlap function

$$O^n(\phi) = \sum_{\sigma \in \pm} \sigma \int d^2 r_2 \ldots d^2 r_N (v_0^{\sigma,n}(\vec{r}_2, ., \vec{r}_N))^* v_{\sigma 1}^{\sigma,n}(\vec{r}_2, ., \vec{r}_N) \tag{9}$$

we get for S_d^n by using (1)

$$S_d^n(\phi) = N\frac{\pi}{m}\phi \left(\left(\frac{\phi}{2} - 1\right) O^n(\phi) + \left(\frac{\phi - 1}{2}\right)(O^n(\phi))^* \right). \tag{10}$$

This is a term which is in general not zero.

One can interpret formula (6) as the energy variation of the electron system caused by an induced electric field originating from a variation of the magnetic string field. The first term in (6) originates from the force of the induced electric field on the electrons outside the strings and the last term from the force inside the magnetic strings. This is an astonishing result because from (5) we see that the electron density at the positions of the magnetic strings is zero. Nevertheless, one can have a finite current density at the positions of the strings resulting in this additional term. We should mention that this additional term was erroneously omitted in[4].

A. Space inversion symmetry of the system

In the following, we will show by using the space inversion symmetry of the two string system that the energy spectrum does not depend on the string distance d for $d > 0$. To achieve this result we will first express $\partial_d E^n(\phi,d)$ in

(6) as a pure surface term. Due to the space inversion symmetry, we have the following relation for the eigenfunctions of H_{ss}

$$
\begin{aligned}
&\Psi^n_{1-\phi,d}(\vec{r}_1,..,\vec{r}_N) \\
&= \Psi^n_{\phi,d}(-\vec{r}_1,..,-\vec{r}_N)e^{-i\sum_i \arg[\vec{r}_i^-]}e^{+i\sum_i \arg[\vec{r}_i^+]}\,.
\end{aligned}
\tag{11}
$$

Here $\arg[\vec{r}]$ is the angle between $\vec{r}$ and the x axis. The wavefunction $\Psi^n_{1-\phi,d}(\vec{r}_1,..,\vec{r}_N)$ is an eigenfunction of H_{ss} with string strength $1-\phi$, i.e.

$$
E^n(\phi,d) = E^n(1-\phi,d)\,.
\tag{12}
$$

Thus we denote the related states (11) by the same label n. The current expectation value is given by

$$
\langle \hat{\vec{J}}(\vec{r})\rangle^n_{1-\phi,d} = -\langle \hat{\vec{J}}(-\vec{r})\rangle^n_{\phi,d}\,.
\tag{13}
$$

In summary, due to the different signs of the strengths of the two strings we obtain an inversion symmetry of the energy spectrum with respect to the strength $\phi = 0.5$.

By using the symmetry of the wave functions (11) in order to get the corresponding symmetries for the values of the wavefunctions at the strings (5) we obtain for $S^n_d(1-\phi)$ by using (9) and (10)

$$
S^n_d(1-\phi) = N\frac{\pi}{m}(\phi-1)\left(\frac{\phi}{2}O^n(\phi) + \left(\frac{\phi+1}{2}\right)(O^n(\phi))^*\right)\,.
\tag{14}
$$

In the following, we calculate the derive of the state energy $E^n(\phi,d)$ with respect to d. By using $\vec{F}_{\phi,d}(\vec{r}) = \vec{F}_{\phi,d}(-\vec{r})$, $E^n(\phi,d) = E^n(1-\phi,d)$ and (6), (13) we get

$$
\begin{aligned}
\partial_d E^n(\phi,d) &= -\frac{\phi}{1-\phi}(\partial_d \vec{F}_{\phi,d}(\vec{r}))\langle \hat{\vec{J}}(\vec{r})\rangle^n_{1-\phi,d} + S^n_d(\phi) \\
&= -\frac{\phi}{1-\phi}\partial_d E^n(\phi,d) + \frac{\phi}{1-\phi}S^n_d(1-\phi) + S^n_d(\phi)\,,
\end{aligned}
\tag{15}
$$

for $d>0$. With the definition of the overlap function $O^n(\phi)$ (9) and by using (10), (14) we get

$$
\frac{\phi}{1-\phi}S^n_d(1-\phi) + S^n_d(\phi) = -N\frac{2\pi}{m}\phi\,\mathrm{Re}\,[O^n(\phi)]\,.
\tag{16}
$$

Now we insert (16) in (15) to get

$$
\partial_d E^n(\phi,d) = N\frac{2\pi}{m}\phi\,(\phi-1)\mathrm{Re}\,[O^n(\phi)]\,.
\tag{17}
$$

Thus, we obtain a formula which relates the distance dependence of the energy spectrum to the asymptotic values of the wavefunction at the two magnetic strings. When neglecting the surface term on the right hand side which stems from the domain subtleties of the Hamilton operator H_{ss} mentioned above we obtain the invariance of the energy spectrum as a function of the string distance d. In the following, we will show that this term is in fact zero. This will be done by deriving a virial theorem for the two string system.

B.   Generalized virial theorem for the two string system

In order to derive this virial theorem we consider the operator $\vec{\Pi}_i = (-i\vec{\nabla}_i + \vec{A} + \phi\vec{f}(\vec{r}_i^-) - \phi\vec{f}(\vec{r}_i^+))$ of the ith electron. This operator fulfills the following commutation relation

$$
[\Pi_{x_i},\Pi_{y_i}] = i(B - \phi\,\delta(\vec{r}_i^-) + \phi\,\delta(\vec{r}_i^+))\,.
\tag{18}
$$

For $N=1$, the ladder operators $\Pi^+_i = (\Pi_{x,i} + i\Pi_{y,i})/2m$ $(\Pi^-_i = (\Pi_{x_i} - i\Pi_{y_i})/2m)$, when acting on an eigenfunction of H_{ss}, raise (lower) the eigenvalue (remember that the eigenfunctions of H_{ss} are zero at the strings for $\phi \neq 0$). This is well known for a single electron in a homogeneous magnetic field without strings. However, a generalization of this procedure in the presence of strings is rather difficult. This is due to the fact that the wavefunctions created by acting

with the ladder operators Π_i^+ and Π_i^- on an eigenfunction need not be in the domain $\mathcal{D}[H_{\phi,d}]$ (2) of H_{ss} for $\phi \neq 0$. This can be shown easily the known ground state wavefunctions of the one and two string systems[4].

In order to express the surface term on the right hand side of equation (17) as a function of the energy $E^n(\phi, d)$ we consider the operator $\sum_{i=1}^N \{\vec{r}_i, \vec{\Pi}_i\}$ where $\{\cdot, \cdot\}$ is the anti-commutator. Then, by using (18) and the homogeneity of the interaction function V^{ee} we obtain

$$i \left[H_{ss}, \sum_{i=1}^N \left\{ \vec{r}_i, \vec{\Pi}_i \right\} \right] = 4H_{ss} + (\gamma - 2)\alpha \sum_{i \neq j} V^{ee}(\vec{r}_i - \vec{r}_j)$$

$$+ \frac{1}{m} \left[\sum_{i=1}^N B \left[\vec{r}_i \overset{\times}{,} \vec{\Pi}_i \right] + \left[\frac{\vec{r}_i}{2} \overset{\times}{,} \left\{ \phi\,\delta(\vec{r}_i^+) - \phi\,\delta(\vec{r}_i^-), \vec{\Pi}_i \right\} \right] \right]. \tag{19}$$

Here $\vec{A} \times \vec{B} - \vec{B} \times \vec{A}$ is denoted by $[\vec{A} \overset{\times}{,} \vec{B}]$ for two vector operators $\vec{A}$, $\vec{B}$.

In the following, we will calculate the expectation values of the various operators in (19) with respect to the eigenfunction $\Psi_{\phi,d}^n$. The expectation values of the two operators which contain only surface terms are given by

$$i \langle [H_{ss}, \sum_{i=1}^N \{\vec{r}_i, \vec{\Pi}_i\}] \rangle_{\phi,d}^n = N \frac{4\pi d}{m} \phi(1-\phi) \mathrm{Re}[O^n(\phi)], \tag{20}$$

$$\frac{1}{2m} \sum_{i=1}^N \left\langle \left[\vec{r}_i \overset{\times}{,} \left\{ \phi\,\delta(\vec{r}_i^+) - \phi\,\delta(\vec{r}_i^-), \vec{\Pi}_i \right\} \right] \right\rangle_{\phi,d}^n = N \frac{2\pi d}{m} \phi(1-2\phi) \mathrm{Re}[O^n(\phi)]. \tag{21}$$

Now we have everything that is required for the derivation of the energy formula. With the help of (1), (17), (20) and (21) we obtain the following differential equation for the energy function $E^n(\phi, d)$

$$\frac{d}{(\phi-1)} \frac{\partial}{\partial d} E^n(\phi, d) = 4 \left(1 - B\frac{\partial}{\partial B} + \frac{(\gamma-2)\alpha}{2} \frac{\partial}{\partial \alpha} \right) E^n(\phi, d) \tag{22}$$

This is a remarkable result stating that the energy is a homogeneous function of the possibly rescaled variables α, B and d. We now come to the conclusion that the energy function does not depend on the string distance d for $d > 0$. This can be seen by the use of the relation (12). From this mirror symmetry relation of the spectrum we get that $E^n(\phi, d)$ fulfills also the differential equation (22) in the case of the substitution $\phi \to (1-\phi)$ in the left hand side of (22). Thus we obtain $(\partial/\partial d)E^n(\phi, d)/(\phi - 1) = -(\partial/\partial d)E^n(\phi, d)/\phi$. This equation can only be fulfilled when $(\partial/\partial d)E^n(\phi, d) = 0$ identically. Summarizing, we obtain:

$$\frac{\partial}{\partial d} E^n = 0\,, \tag{23}$$

$$E^n - B\frac{\partial}{\partial B}E^n + \frac{(\gamma-2)\,\alpha}{2}\,\frac{\partial}{\partial \alpha}E^n = 0\,. \tag{24}$$

This is the main result of this section. It is valid only in the case of a homogeneous interaction potential V^{ee}.

The first equation states that the energy spectrum does not depend on the string separation distance for $d > 0$. We get the constraint $d > 0$ because we used the asymptotic formula (5) for the eigenfunctions in the derivation above. This formula is only valid for a finite positive string distance because the eigenfunctions and eigenvalues are not continuous in general at $d = 0$. Nevertheless, we will show in the following section that the ground state energy as well as the ground state degeneracy are continuous as a function of d at $d = 0$.

The differential equation (24) states that $E^n(\phi, d)$ is a homogeneous function of the magnetic field B and the interaction coupling constant $\alpha^{(2-\gamma)/2}$. For $\phi = 0$ and $B = 0$ this equation is the well known quantum mechanical version of the virial theorem[16].

The above results for the homogeneous two particle interaction potential V^{ee} can be generalized. It is easily seen from the derivation above that (23) and (24) are also valid for homogeneous many particle potentials $\alpha V(\vec{r}_1, .., \vec{r}_N)$. Furthermore, (23) and (24) can be generalized to the more general case of an inversion symmetrical potential, i.e. $\alpha V(\vec{r}_1, \ldots, \vec{r}_N) = \alpha V(-\vec{r}_1, \ldots, -\vec{r}_N)$ by substituting $(\gamma/2 - 1)\alpha(\partial/\partial\alpha)E^n \to -\alpha \langle (V(\vec{r}_1, .., \vec{r}_N) + \sum_{i=1}^N (\vec{r}_i \cdot \vec{\nabla}_i)V(\vec{r}_1, .., \vec{r}_N)/2) \rangle_{\phi,d}^n$ in the last term on the right hand side of (24). We should point out that the physically relevant interaction potential of the jellium system is exactly of that form.

III. GROUND STATE ENERGY FOR SMALL STRING SEPARATION

In this section we will show the continuity of the ground state energy for small string separations. In the following we briefly outline the strategy of the proof deferring the details to appendix A.

Given a (degenerate) ground state wavefunction of the electron system without strings (a ground state wavefunction of $H_{ss}(0,0)$), we construct wavefunctions which depend on the string separation d. These wavefunctions have expectation values with respect to the Hamiltonian $H_{ss}(\phi, d)$ which converge towards the ground state energy of $H_{ss}(0,0)$ for $d \to 0$. By using the Rayleigh-Ritz variational principle we obtain $\lim_{d \to 0}(E_{\phi,d}^{\mathrm{GS}} - E_{0,0}^{\mathrm{GS}}) \leq 0$ (A10) where $E_{\phi,d}^{\mathrm{GS}}$ is the ground state energy of H_{ss}. A rigorous proof of the continuity of the ground state energy for small string separations, requires the proof of the reverse, too. Given a ground state wave function of $H_{ss}(\phi, d)$ we will construct wavefunctions (depending on string distance d) for which we calculate the expectation value with respect to the Hamiltonian $H_{ss}(0,0)$. We will show that the difference of this expectation value and the ground state energy of $H_{ss}(\phi, d)$ converges to zero for $d \to 0$. By using the Rayleigh-Ritz variational principle we obtain $\lim_{d \to 0}(E_{\phi,d}^{\mathrm{GS}} - E_{0,0}^{\mathrm{GS}}) \geq 0$ (A12).

By combining (A10) and (A12) we obtain that the ground state energy of $H_{ss}(\phi, d)$ is continuous at $d = 0$, i.e.

$$\lim_{d \to 0}(E_{\phi,d}^{\mathrm{GS}} - E_{0,0}^{\mathrm{GS}}) = 0 \ . \tag{25}$$

From the explicit construction carried out in appendix A one further obtains the continuity of the ground state degeneracy at $d = 0$.

IV. GROUND STATE ENERGY WITHOUT RESTRICTIONS ON THE STRING SEPARATION

In section II, we have shown that the energy spectrum does not depend on the string separation. In section III, the continuity of the ground state energy at $d = 0$ has been shown. Using Eq. (23) and Eq. (25) we obtain that the ground state energy is the same for the system with or without the two strings, i.e.

$$E_{\phi,d}^{\mathrm{GS}} = E_{0,0}^{\mathrm{GS}} \ , \tag{26}$$

Furthermore, we obtain from the remark below (25) that the ground state degeneracy is also the same.

We should mention that this result is to our understanding by no means trivial which can be seen by considering the ground state energy of an interacting electron system in the background of one string. Let us suppose that the ground state degeneracy is lifted by the interaction between the electrons for the system without the string. This ground state wavefunction is changed when adiabatically switching on the magnetic string. During this process the angular momentum of the wavefunction in z-direction remains constant. When the string reaches one flux quantum the ground state of this system is connected with the ground state of the system without the string by a phase transformation. Thus, the angular momentum of the two ground states for the system with zero flux quantum and one flux quantum differs by $N\hbar$. This means that the ground states of the system with zero string flux and with string flux one are not connected in the energy spectrum as a function of the string strength. Then we obtain a crossing of the energy values of these two states at values between zero and one. This crossing should in general happen at an energy above the ground state energy of the system without the magnetic string. Thus, we obtain that the ground state energy is in general not the same with or without the string. The main difference between the one string system and the system with two opposite strings considered in this paper is that the net flux is zero for the two string system. Thus, one could suppose that the results above are true in general for a many string system with resulting net zero flux. This is an open question which can not be answered by the symmetry considerations carried out in this paper.

At last we compare the results derived in this paper with the corresponding results for the non-interacting ($N = 1$) two string system in a homogeneous magnetic field considered in Ref. 4. There we obtain that only the energy of the wavefunction which is constant on both strings increases when switching on the magnetic strings. The energies of the rest of the ground states are constant. Due to the infinite degeneracy of the ground state we obtain the correspondence between the results in this paper and in Ref. 4 concerning the ground state energy and the ground state degeneracy of the two string system. The increase of the energy of one of the ground states when switching on the strings is a pure edge effect which is linked to the property that the degeneracy of the non-interacting system is proportional to its area.

V. CONCLUSION

We discuss in this paper the problem of the ground state energy calculation for an infinitely extended system of interacting electrons in a homogeneous magnetic field and in the background of two magnetic strings with opposite

flux. We treat at first an interaction potential for the electrons which is homogeneous. By using the inversion symmetry of the two string system and further some commutator algebra we obtain two equations. The first equation (23) states that the energy spectrum does not depend on the separation distance of the two strings for strictly positive string separation. The second equation (24) has the same shape as the virial theorem equation for the system without strings. It contains only differentials with respect to the magnetic field and the interaction strength. This equation gives evidence on the homogeneity of the energy function as a function of the rescaled magnetic field and interaction strength. Next, we showed by using the Rayleigh Ritz variational principle that the ground state energy and the ground state degeneracy are continuous functions at zero string separation.

By using these properties, we obtain the main result of this paper: The ground state energy of the interacting system is the same with and without the two magnetic strings. This is an answer to a question that arose in the context of determining a theory for the fractional quantum Hall effect with a finite Green's function[4] but will also be useful in other fields of physics, as for example in the dislocation theory of solids.

We should point out that the results concerning the independence of the energy spectrum with respect to the string separation and the invariance of the ground state energy are also valid for the more general class of inversion symmetrical many particle interaction potentials for indistinguishable particles. For example, the two particle interaction potentials which depend on the distance between the electrons are of that form. Furthermore, we obtain the validity of the results also for the jellium model. We can generalize our findings to the case of bosons.

APPENDIX A: PROOF OF THE CONTINUITY OF THE GROUND STATE ENERGY FOR SMALL STRING SEPARATION

In this appendix, we give a formal proof that the the ground state energy and the ground state degeneracy are continuous functions of d for $d \to 0$ following the strategy outlined in section III .

1. Proof of $\lim_{d\to 0}(E_{\phi,d}^{\mathrm{GS}} - E_{0,0}^{\mathrm{GS}}) \leq 0$

We first construct approximate ground state wavefunctions of $H_{ss}(\phi,d)$ from the ground state of $H_{ss}(0,0)$. To this end, we define the following auxiliary function

$$
P_{\phi_d,d} := \begin{cases}
(r^-)^{\phi_d} & \text{for} \quad x \leq -d/2 \,, \\
\frac{y^2}{(|y|+|x_2^-|)^{2-\phi_d}} & \text{for} \quad -d/2 \leq x < 0 \,, \\
\frac{y^2}{(|y|+|x^+|)^{2-\phi_d}} & \text{for} \quad 0 \leq x < +d/2 \,, \\
(r^+)^{\phi_d} & \text{for} \quad d/2 \leq x \,.
\end{cases}
\tag{A1}
$$

which is continuous on $\mathbb{R}^2$. Let $\Psi_{0,0}^{\mathrm{GS}}$ be an arbitrary ground state wavefunction of $H_{ss}(0,0)$. The corresponding approximate ground state of $H_{ss}(0,0)$ is given by

$$
\Psi_{\phi,d} := \frac{1}{N(\phi,d)} \left(\prod_{i=1}^{N} e^{i\phi \arg[\vec{r}_i^+]} e^{-i\phi \arg[\vec{r}_i^-]} P_{\phi_d,d}(\vec{r}_i) \right) \Psi_{0,0}^{\mathrm{GS}}
\tag{A2}
$$

with $N(\phi,d)$ a normalization factor defined by $\int d^2 r |\Psi_{\phi,d}(\vec{r})|^2 = 1$ and

$$
\phi_d = \frac{1}{\sqrt{|\ln(d)|}} \,.
\tag{A3}
$$

Due to the non-differentiability of $\Psi_{\phi,d}$ on the axes $x = -d/2$, $x = 0$ and $x = d/2$ we get that $\Psi_{\phi,d}$ is not in the domain of $H_{ss}(\phi,d)$. Nevertheless, $\Psi_{\phi,d}$ is continuous on $\mathbb{R}^2$. It can be shown that the wavefunction $\Psi_{\phi,d}$ can be smoothed in a way that its expectation value with respect to $H_{ss}(\phi,d)$ remains unchanged up to an infinitesimal amount and lies in the domain of $H_{ss}(\phi,d)$. By definition, it is immediately clear that $\lim_{d\to 0} N(\phi,d) = 1$. In the following, we show that $\lim_{d\to 0}\langle \Psi_{\phi,d}|H_{ss}(\phi,d)|\Psi_{\phi,d}\rangle = E_{0,0}^{\mathrm{GS}}$. We restrict the discussion to the one particle case in order to simplify the notation. The following results can be generalized easily to the many particle case. With the help of

$$
T_1 := \frac{1}{N(\phi,d)} \frac{1}{2m} \langle \Psi_{0,0}^{\mathrm{GS}}|(\vec{\nabla} P_{\phi_d,d}(\vec{r}))(\vec{\nabla} P_{\phi_d,d}(\vec{r}))|\Psi_{0,0}^{\mathrm{GS}}\rangle \,,
\tag{A4}
$$

$$T_2 := -\frac{1}{N(\phi,d)}\frac{1}{2m}\langle\Psi_{0,0}^{\text{GS}}|P_{\phi_d,d}(\vec{r})(\vec{\nabla}P_{\phi_d,d}(\vec{r}))\vec{\nabla} + \vec{\nabla}P_{\phi_d,d}(\vec{r})(\vec{\nabla}P_{\phi_d,d}(\vec{r}))|\Psi_{0,0}^{\text{GS}}\rangle\,, \tag{A5}$$

$$T_3 := \frac{1}{N(\phi,d)}\frac{1}{i}\frac{1}{m}\langle\Psi_{0,0}^{\text{GS}}|P_{\phi_d,d}(\vec{r})(\vec{A}(\vec{r})(\vec{\nabla}P_{\phi_d,d}(\vec{r}))|\Psi_{0,0}^{\text{GS}}\rangle \tag{A6}$$

we obtain $\langle\Psi_{\phi,d}|H_{ss}(\phi,d)|\Psi_{\phi,d}\rangle = E_{0,0}^{\text{GS}} + T_1 + T_2 + T_3$. Thus, we have to show that $\lim_{d\to 0}(T_1 + T_2 + T_3) = 0$.

The expectation values in (A4)-(A6) stand for a two dimensional integration over the plane. In the following, we will show that $\lim_{d\to 0} T_1 = 0$ is fulfilled separately for the four different integration areas $x < -d/2$, $-d/2 < x < 0$, $0 < x < d/2$ and $d/2 < x$. It is immediately clear from the definitions (A1)-(A6) that $\lim_{d\to 0} T_1 = 0$ is fulfilled for the integration area $x < -d/2$ and $x > d/2$. Now we will show first that $\lim_{d\to 0} T_1 = 0$ is fulfilled for the integration area $-d/2 < x < 0$ and $y > 0$. In this area, we get

$$(\vec{\nabla}P_{\phi_d,d})^2 = \frac{2(2 + \phi_d^2 - 2\phi_d)y^4}{(y + |x^-|)^{6-2\phi_d}} + \frac{4\phi_d\, x^-y^3 + 4(x^-)^2y^2}{(y + |x^-|)^{6-2\phi_d}}\,. \tag{A7}$$

Since $\Psi_{0,0}^{\text{GS}}$ is continuous differentiable it is sufficient to show that $\lim_{d\to 0}\int_{-d/2<x<0} d^2r(\vec{\nabla}P_{\phi_d,d}(\vec{r}))^2 = 0$. As an example we show the asymptotic vanishing of the first term in (A7). The vanishing of the last term in (A7) works similarly. By carrying out the integration we get for the first term

$$\int_{-d/2<x<0,y>0} d^2r\left[(\vec{\nabla}P_{\phi_d,d})^2\right]_1 = -\frac{(2 + \phi_d^2 - 2\phi_d)}{(5 - 2\phi_d)\phi_d}\left[(y_0 + d/2)^{2\phi_d} - (d/2)^{2\phi_d} - (y_0)^{2\phi_d} - 2\phi_d\int_0^{y_0} dy\,\frac{(y + d/2)^4 - y^4}{(y + d/2)^{5-2\phi_d}}\right]\,. \tag{A8}$$

Here y_0 is the y-coordinate of the upper edge of the sample (without loss of generality we assume a square sample with center at the origin). By a binomial expansion of the first term in the square brackets on the right hand side of (A8) and the definition of ϕ_d in (A3) we find that the sum of the first three terms on the right hand side is zero for $d \to 0$. The last term in the square brackets in (A8) converges by a factor ϕ_d faster to zero than the sum of the first three terms. By carrying out a similar analysis for the last term in (A7) we get $\lim_{d\to 0}\int_{-d/2<x<0,y>0} d^2r(\vec{\nabla}P_{\phi_d,d})^2 = 0$. Due to the mirror symmetry according to the y-axis and x-axis of $P_{\phi_d,d}$ we then find also that

$$\lim_{d\to 0}\int_{-d/2<x<d/2} d^2r\,(\vec{\nabla}P_{\phi_d,d})^2 = 0 \tag{A9}$$

and thus $\lim_{d\to 0} T_1 = 0$ for $-d/2 < x < d/2$. At last, we have to show that $\lim_{d\to 0}(T_2 + T_3)$. It is easily seen from the definitions (A5) and (A6) that T_2 and T_3 converge much faster to zero than T_1. Summarizing, we get $\lim_{d\to 0}(T_1 + T_2 + T_3) = 0$.

Now we obtain by the help of $\lim_{d\to 0}\langle\Psi_{\phi,d}|H_{ss}(\phi,d)|\Psi_{\phi,d}\rangle = E_{0,0}^{\text{GS}}$ shown above and the Rayleigh-Ritz variational principle that

$$\lim_{d\to 0}(E_{\phi,d}^{\text{GS}} - E_{0,0}^{\text{GS}}) \le 0\,. \tag{A10}$$

Furthermore, we obtain from the definition of $\Psi_{\phi,d}$ (A2) that the number of orthogonal states with an energy lower than or equal to $E_{0,0}^{\text{GS}}$ is larger than or equal to the degeneracy of the ground state energy $E_{0,0}^{\text{GS}}$ for $d \to 0$.

2. The proof of $\lim_{d\to 0}(E_{\phi,d}^{\text{GS}} - E_{0,0}^{\text{GS}}) \ge 0$

Now we have to show the reverse, i.e. given the (degenerate) ground state wavefunction $\Psi_{\phi,d}^{\text{GS}}$ of energy $E_{\phi,d}^{\text{GS}}$, we construct an approximate ground state wavefunction $\Psi_{0,0}$ for $H_{ss}(0,0)$ (1). This is done by defining $\Psi_{0,0}$ as in (A2) with substitutions $\Psi_{0,0}^{\text{GS}} \to \Psi_{\phi,d}^{\text{GS}}$, $\arg[\vec{r_i^-}] \to -\arg[\vec{r_i^-}]$ and $\arg[\vec{r_i^+}] \to -\arg[\vec{r_i^+}]$. Then we have to show that $\lim_{d\to 0}\langle\Psi_{0,0}|H_{ss}(0,0)|\Psi_{0,0}\rangle - E_{\phi,d}^{\text{GS}} = 0$. It is clear that the proof works analogously to the proof of (A10).

The ground state energy $E_{\phi,d}^{\text{GS}}$ is given by

$$E_{\phi,d}^{\text{GS}} = \int d^2r[(-i\vec{\nabla} + \vec{A}(\vec{r}) + \phi\vec{f}(\vec{r^-}) - \phi\vec{f}(\vec{r^+}))\Psi_{\phi,d}^{\text{GS}}]^* [(-i\vec{\nabla} + \vec{A}(\vec{r}) + \phi\vec{f}(\vec{r^-}) - \phi\vec{f}(\vec{r^+}))\Psi_{\phi,d}^{\text{GS}}]\,. \tag{A11}$$

The energy expectation value $\langle\Psi_{0,0}|H_{ss}(\phi,d)|\Psi_{0,0}\rangle$ is given by $E_{\phi,d}^{\text{GS}} + \tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3$ where $\tilde{T}_1$, $\tilde{T}_2$ is given by T_1, T_2 (A4), (A5) with the substitution $\Psi_{00}^{\text{GS}} \to \Psi_{\phi,d}^{\text{GS}}$. $\tilde{T}_3$ is given by T_3 (A6) with the substitutions $\Psi_{00}^{\text{GS}} \to \Psi_{\phi,d}^{\text{GS}}$ and

$\vec{A}(\vec{r}) \rightarrow \vec{A}(\vec{r}) + \phi\vec{f}(\vec{r}^{-}) - \phi\vec{f}(\vec{r}^{+})$. In order to show the inequality $\lim_{d\rightarrow 0}(E^{\text{GS}}_{\phi,d} - E^{\text{GS}}_{0,0}) \geq 0$ we have to show that $\lim_{d\rightarrow 0}(\tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3) = 0$. This can be shown rather easily by using (A9), (A11), and further the well known Schwarz inequality $|\int_A d^2 r f^*(\vec{r})g(\vec{r})|^2 \leq \int_A d^2 r |f(\vec{r})|^2 \cdot \int_A d^2 r |g(\vec{r})|^2$ valid for $\mathbb{L}^2$ wavefunctions f and g in the area A.

Summarizing, we obtain

$$\lim_{d\rightarrow 0}(E^{\text{GS}}_{\phi,d} - E^{\text{GS}}_{0,0}) \geq 0 \,. \tag{A12}$$

Furthermore, we obtain that the number of orthogonal states with an energy lower or equal to $E^{\text{GS}}_{\phi,d}$ is larger than or equal to the degeneracy of the ground state energy $E^{\text{GS}}_{\phi,d}$ for $d \rightarrow 0$.

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[*] Electronic address: dietel@physik.fu-berlin.de
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