Problem Set 2
Quantum Field Theory and Many Body Physics (SoSe2018)

Due: Thursday, May 3, 2018 before the beginning of the class

In this problem set, we continue our discussion of Gaussian integrals by introducing the concept of generating function(al)s of moments and cumulants which are widely used in statistics and quantum field theory. We also fill in some gaps in our treatment of the Jordan-Wigner transformation and discuss further applications of this technique.

Problem 1: Cumulant expansion and generating functionals (10+10+5+5 points)

In this problem, we want to discuss some basics of generating functions for probability distributions and generalize this concept to field theories. Generating functions are a standard tool in probability theory. Consider a random variable x with probability distribution $P(x)$ and denote the corresponding averages by $\langle \ldots \rangle$. Then, the moment generating function

$$\mathcal{G}(J) = \langle e^{Jf(x)} \rangle \tag{1}$$

succinctly summarizes all moments $\langle [f(x)]^n \rangle$ $(n = 0, 1, 2, \ldots)$ of some function $f(x)$. Indeed,

$$\langle [f(x)]^n \rangle = \left. \frac{\mathrm{d}^n}{\mathrm{d}J^n} \mathcal{G}(J) \right|_{J=0} \tag{2}$$

or

$$\mathcal{G}(J) = \sum_{n=0}^{\infty} \frac{J^n}{n!} \langle [f(x)]^n \rangle. \tag{3}$$

For the special case of $f(x) = x$, the moments are just the averages $\langle x^n \rangle$.

Instead of the moments, it is often useful to characterize the probability distribution through its cumulants. Examples are the average $C_1 = \langle f(x) \rangle$, which is the first cumulant, or the variance $C_2 = \langle [f(x)]^2 \rangle - \langle f(x) \rangle^2$, which is the second cumulant. It turns out that the entire series of cumulants C_n is generated by the cumulant generating functional

$$\mathcal{W}(J) = \ln \mathcal{G}(J) = \ln \langle e^{Jf(x)} \rangle \tag{4}$$

through

$$C_n = \left. \frac{\mathrm{d}^n}{\mathrm{d}J^n} \mathcal{W}(J) \right|_{J=0} \tag{5}$$

and

$$\mathcal{W}(J) = \sum_{n=1}^{\infty} \frac{J^n}{n!} C_n. \tag{6}$$

(a) Give explicit expressions for the first four cumulants C_1, C_2, C_3, C_4 in terms of the moments of $f(x)$. The third cumulant is known as skewness, the fourth as kurtosis.

(b) Compute all cumulants of x for a Gauss distribution

$$P(x) = \frac{1}{\sqrt{2\pi\sigma}} \, \mathrm{e}^{-\frac{x^2}{2\sigma}}. \tag{7}$$

and all cumulants of n ($n = 1, 2, 3, \ldots$) for the Poisson distribution

$$P(n) = \tfrac{1}{n!}\, \mathrm{e}^{-\lambda}\, \lambda^n, \tag{8}$$

where $n = 0, 1, 2, ...$ You should do this by explicitly computing the cumulant generating functions for these distributions.

(c) Now consider a multivariate complex Gaussian distribution as introduced in problem 3 of problem set 1,

$$P[\boldsymbol{\phi}] = \frac{\exp\left\{-\boldsymbol{\phi}^\dagger \mathrm{M} \boldsymbol{\phi}\right\}}{\int [\mathrm{d}\boldsymbol{\phi}][\mathrm{d}\boldsymbol{\phi}^*] \exp\left\{-\boldsymbol{\phi}^\dagger \mathrm{M} \boldsymbol{\phi}\right\}} \tag{9}$$

and introduce the moment generating functional

$$\mathcal{G}[\mathbf{J}] = \langle \exp\{\mathbf{J}^\dagger \boldsymbol{\phi} + \boldsymbol{\phi}^\dagger \mathbf{J}\}\rangle \tag{10}$$

as well as the cumulant generating functional

$$\mathcal{W}[\mathbf{J}] = \ln\langle \exp\{\mathbf{J}^\dagger \boldsymbol{\phi} + \boldsymbol{\phi}^\dagger \mathbf{J}\}\rangle. \tag{11}$$

Compute both of these generating functions explicitly (i.e., perform the average). Use this to compute the second moment $\langle \phi_i \phi_j^* \rangle$ and the second cumulant $\langle \phi_i \phi_j^* \rangle - \langle \phi_i \rangle \langle \phi_j^* \rangle$.

(d) Now consider the slightly modified multivariate complex Gaussian distribution,

$$P[\boldsymbol{\phi}] = \frac{\exp\left\{-(\boldsymbol{\phi} - \boldsymbol{\phi}_0)^\dagger \mathrm{M} (\boldsymbol{\phi} - \boldsymbol{\phi}_\mathbf{0})\right\}}{\int [\mathrm{d}\boldsymbol{\phi}][\mathrm{d}\boldsymbol{\phi}^*] \exp\left\{-(\boldsymbol{\phi} - \boldsymbol{\phi}_0)^\dagger \mathrm{M} (\boldsymbol{\phi} - \boldsymbol{\phi}_0)\right\}} \tag{12}$$

with some fixed $\boldsymbol{\phi}_0$. Compute the generating functions as defined in (c). Use your result to obtain the first and second moments and cumulants. (The first moment and cumulant are just the average $\langle \phi_i \rangle$).

Problem 2: Jordan-Wigner transformation (10+15 points)

A single fermionic state can be empty or occupied. Similarly, a spin-1/2 can point either up or down. This suggests that one might be able to map a spin-1/2 degree of freedom to a fermion mode. However, a little thought reveals that this is not so simple. The problem is that operators corresponding to different spins commute while different fermion operators anticommute. In this problem, we will establish that one can in fact find an exact and useful general mapping between spins and fermions in *one dimension* by attaching an additional string operator to a fermion. This is known as Jordan-Wigner transformation.

(i) Let's start with a single spin-1/2 degree of freedom and write the spin operator in terms of a fermion operator. The spin operator is defined by

$$S^j = \frac{1}{2} \sigma^j, \tag{13}$$

where $j = x, y, z$ and σ^j denotes a Pauli matrix. The spin operators satisfy the angular-momentum algebra

$$[S^j, S^k] = i\epsilon^{jkl} S^l \tag{14}$$

and the anticommutation relations

$$\{S^j, S^k\} = \frac{1}{2} \delta^{jk}. \tag{15}$$

We will also use the usual raising and lowering operators $S^\pm = S^x \pm i S^y$.

Now we identify the spin-down state $|\downarrow\rangle$ (satisfying $S^z|\downarrow\rangle = -(1/2)|\downarrow\rangle$) with the vacuum state $|0\rangle$ for a fermion operator f (i.e., $f|0\rangle = 0$) and the spin-up state $|\uparrow\rangle$ with the occupied fermion state $|1\rangle = f^\dagger|0\rangle$. Show that we can make the identifications

$$S^x = \frac{1}{2}\left(f + f^\dagger\right) \tag{16}$$

$$S^y = \frac{i}{2}\left(f - f^\dagger\right) \tag{17}$$

$$S^z = f^\dagger f - \frac{1}{2} \tag{18}$$

$$S^+ = f^\dagger \tag{19}$$

$$S^- = f. \tag{20}$$

Confirm that these operators indeed satisfy the commutation and anticommutation relations of the spin operators.

(ii) Now consider a one-dimensional lattice with sites labeled by $j = \ldots -2, -1, 0, 1, 2 \ldots$. Define a spin operator $\mathbf{S}_j$ and a fermion operator f_j on every site. We can no longer directly use the previous mapping between spin and fermion because the spin operators belonging to different sites commute while the corresponding fermion operators anticommute. To fix this, consider the string operator

$$e^{i\phi_j} = e^{i\pi \sum_{k<j} n_k}, \tag{21}$$

where $n_k = f_k^\dagger f_k$. Explain why this is a hermitian operator. Now show that a spin can be thought of as a fermion with an attached string operator by verifying that the Jordan-Wigner transformation

$$S_j^z = f_j^\dagger f_j - \frac{1}{2} \tag{22}$$

$$S_j^+ = f_j^\dagger e^{i\phi_j} \tag{23}$$

$$S_j^- = f_j e^{-i\phi_j} \tag{24}$$

preserves the (anti)commutation relation on each site and correctly yields commuting spin operators on different sites. You may find it helpful to first show that the string operator anticommutes with each fermion operator to the left of its open end and commutes with fermion operators at or to the right of its open end,

$$\{f_k, e^{i\phi_j}\} = 0 \quad ; \quad k < j \tag{25}$$

$$[f_k, e^{i\phi_j}] = 0 \quad ; \quad k \geq j \tag{26}$$

Problem 3: Quantum XXZ model (5+10+10 points)

In this problem, we want to use the Jordan-Wigner transformation to discuss the XXZ Hamiltonian in one dimension,

$$H = -\sum_j \{J[S_j^x S_{j+1}^x + S_j^y S_{j+1}^y] + J_z S_j^z S_{j+1}^z\}. \tag{27}$$

For the isotropic case with $J_z = J$, this is known as the quantum Heisenberg model. For $J_z = 0$, the model becomes the quantum xy model. For $J, J_z > 0$, the model describes a ferromagnet in which it is energetically favorable for neighboring spins to be parallel. Antiferromagnetic coupling corresponds to $J, J_z < 0$. We can also write this Hamiltonian as

$$H = -\sum_j \left\{\frac{J}{2}[S_j^+ S_{j+1}^- + S_{j+1}^+ S_j^-] + J_z S_j^z S_{j+1}^z\right\}. \tag{28}$$

(i) Show that the Jordan-Wigner transformation maps the XXZ Hamiltonian H to

$$H = -\frac{J}{2}\sum_j [f^\dagger_{j+1} f_j + f^\dagger_j f_{j+1}] - J_z \sum_j (n_j - \frac{1}{2})(n_{j+1} - \frac{1}{2}). \tag{29}$$

This Hamiltonian describes spinless fermions in one dimension with nearest-neighbor hopping and interactions.

(ii) First consider the xy model with $J_z = 0$. We see that this model maps to non-interacting fermions and can thus be readily solved exactly. Specifically, the xy model maps to the fermion Hamiltonian

$$H = -\frac{J}{2}\sum_j [f^\dagger_{j+1} f_j + f^\dagger_j f_{j+1}]. \tag{30}$$

Show that this simple tight-binding Hamiltonian can be diagonalized by transforming to the momentum representation (translation invariance),

$$H = \sum_k \epsilon_k \, c^\dagger_k c_k \tag{31}$$

with $\epsilon_k = -J\cos k$ and $k \in [-\pi, \pi]$. Give explicit expressions for the operators c_k. Discuss the ground state of the fermion model and its excitation spectrum. (Be sure to notice that some of the single-particle energies are negative!) Use this to compute the ground state energy of the original xy model and to explain that its excitation spectrum is characterized by a linear magnon dispersion. Does the ground state have a spontaneous magnetization?

(iii) Next consider the isotropic Heisenberg model. In this case, we cannot easily find an exact solution because of the nearest-neighbor interaction between the fermions. Nevertheless, we can discuss the properties of this model approximately. Let us first neglect the interaction term $\sum_j n_j n_{j+1}$ and discuss the resulting non–interacting Hamiltonian. First find the single-particle spectrum of this non-interacting problem. You should find that the ground state corresponds to the fermion vacuum which is equivalent to all spins pointing in the spin-down direction. Thus, we actually find a ferromagnetic ground state in this model. Show that the magnon dispersion above this ground state has a quadratic dispersion (unlike the xy model which had a linear dispersion). Now return to the interacting Hamiltonian and explain why one may be tempted to conclude that the interaction term is weak and that it may be a good approximation to neglect it.