

Problem Set 8

Quantum Field Theory and Many Body Physics (SoSe2018)

Due: Thursday June 14, 2018 at the beginning of the lecture

In this problem set, we study noninteracting fermions in a disordered potential. Approximating the electron liquid as non-interacting fermions is the simplest (but often very successful) description of metals or semiconductors. Incorporating disorder into this theory is clearly important as any real metal or semiconductor will not be a perfect single crystal but rather full of defects. In practice, disorder effects are important to understand the transport properties (such as the conductivity) of metals and semiconductors. Disorder can dramatically change the nature of the eigenstates. While the eigenstates of perfect crystals are Bloch states that are extended throughout the entire crystal, eigenstates in disordered systems can be Anderson localized, with the wavefunction falling off exponentially from their center with a localization length ξ . (P.W. Anderson, *Absence of Diffusion in Certain Random Lattices*, Phys. Rev. **109**, 1492 (1958); Anderson received the Nobel Prize in 1977 for this work.) It is believed that the eigenfunctions are always localized in one and two dimensions, even for arbitrarily weak disorder. Such systems would be insulating at zero temperature (Anderson insulator). In three dimensions, the spectrum can support both extended and localized states which are separated in energy by a so-called mobility edge. Such systems can be either metallic or insulating, depending on whether the Fermi energy falls into the region of localized or extended states. Here, we will not be able to discuss these more advanced problems but in principle, these results can be obtained with the methods described in this problem set.

Problem 1: Fermions in a disorder potential

(5+5+5+10 points)

Consider a non-interacting fermion system subject to a potential $V(\mathbf{r})$,

$$H = \int d\mathbf{r} \psi^\dagger(\mathbf{r}) \left(-\frac{\nabla^2}{2m} + V(\mathbf{r}) \right) \psi(\mathbf{r}). \quad (1)$$

(a) Show that the thermal Green function

$$\mathcal{G}(\mathbf{r}\tau, \mathbf{r}'\tau') = \langle \mathcal{T}_\tau \psi(\mathbf{r}, \tau) \psi^\dagger(\mathbf{r}', \tau') \rangle_\pm \quad (2)$$

satisfies the equation of motion

$$(\partial_t + H) \mathcal{G}(\mathbf{r}\tau, \mathbf{r}'\tau') = \delta(\mathbf{r} - \mathbf{r}') \delta(\tau - \tau'), \quad (3)$$

with the single-particle Hamiltonian $H = -\frac{\nabla^2}{2m} + V(\mathbf{r})$.

(b) Consider the thermal Green function $\mathcal{G}_0(\mathbf{r}\tau, \mathbf{r}'\tau')$ of the free particle with $V(\mathbf{r}) = 0$. Show that \mathcal{G} satisfies the Dyson equation

$$\mathcal{G}(\mathbf{r}\tau, \mathbf{r}'\tau') = \mathcal{G}_0(\mathbf{r}\tau, \mathbf{r}'\tau') - \int d\mathbf{r}_1 d\tau_1 \mathcal{G}_0(\mathbf{r}\tau, \mathbf{r}_1\tau_1) V(\mathbf{r}_1) \mathcal{G}(\mathbf{r}_1\tau_1, \mathbf{r}'\tau'), \quad (4)$$

or

$$\mathcal{G} = \mathcal{G}_0 - \mathcal{G}_0 V \mathcal{G} \quad (5)$$

in matrix notation (with $V(\mathbf{r}\tau, \mathbf{r}'\tau') = V(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') \delta(\tau - \tau')$).

(c) Solve the Dyson equation iteratively in powers of V . Show that the result can be represented graphically as

$$\begin{array}{c} \text{1} \quad \text{2} \quad \text{1} \quad \text{2} \quad \text{1} \quad \text{3} \quad \text{4} \quad \text{2} \quad \text{1} \quad \text{3} \quad \text{4} \quad \text{5} \quad \text{6} \quad \text{2} \\ \leftarrow \quad \leftarrow \quad \leftarrow \quad \leftarrow \quad \leftarrow \quad \times \quad \leftarrow \quad \leftarrow \quad \leftarrow \quad \times \quad \leftarrow \quad \times \quad \leftarrow \end{array}$$

where the bold arrow represents the full Green function and the normal arrows stand for the free particle Green function (the internal labels 3, 4, 5, 6 can be dropped). Give the explicit rule how to associate each of these Feynman diagrams with an analytical expression.

(d) Assume that $V(\mathbf{r})$ is a random potential, so that the Hamiltonian might describe a disordered metal in the non-interacting electron approximation. We assume that the random potential is drawn from the (functional) Gaussian probability distribution

$$P[V(\mathbf{r})] = \frac{1}{N} \exp \left[-\frac{1}{2\gamma} \int d\mathbf{r} V^2(\mathbf{r}) \right], \quad (6)$$

where $N = \int [dV(\mathbf{r})] P(V)$ is a normalization constant. Define the generating functional (denoting averages over the disorder potential $V(\mathbf{r})$ by an overbar)

$$W[\rho(\mathbf{r})] = \frac{1}{N} \int [dV(\mathbf{r})] \exp \left[-\frac{1}{2\gamma} \int d\mathbf{r} V^2(\mathbf{r}) + \int d\mathbf{r} \rho(\mathbf{r}) V(\mathbf{r}) \right] = \overline{e^{\int d\mathbf{r} \rho(\mathbf{r}) V(\mathbf{r})}}. \quad (7)$$

Show that

$$W[\rho(\mathbf{r})] = e^{\frac{\gamma}{2} \int d\mathbf{r} \rho^2(\mathbf{r})}. \quad (8)$$

Use this result to show that

$$\overline{V(\mathbf{r})} = 0, \quad (9)$$

$$\overline{V(\mathbf{r})V(\mathbf{r}')} = \gamma\delta(\mathbf{r} - \mathbf{r}'), \quad (10)$$

$$\overline{V(\mathbf{r}_1)V(\mathbf{r}_2)V(\mathbf{r}_3)V(\mathbf{r}_4)} = \gamma\delta(\mathbf{r}_1 - \mathbf{r}_2)\gamma\delta(\mathbf{r}_3 - \mathbf{r}_4) + \gamma\delta(\mathbf{r}_1 - \mathbf{r}_3)\gamma\delta(\mathbf{r}_2 - \mathbf{r}_4) + \gamma\delta(\mathbf{r}_1 - \mathbf{r}_4)\gamma\delta(\mathbf{r}_2 - \mathbf{r}_3). \quad (11)$$

Generally, all correlators of an even number of $V(\mathbf{r})$ are given as the sum over all possible pairings (Wick's theorem).

Problem 2: Impurity-averaged Green function

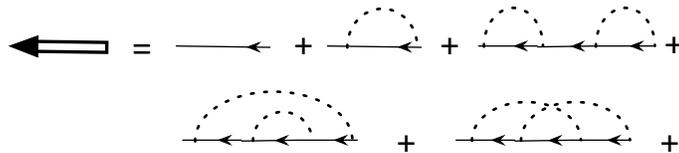
(10+10+5 points)

In this problem, we use the formalism of problem 1 to compute the impurity averaged Green function.

(a) Consider the impurity averaged Green function

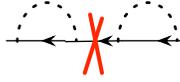
$$\overline{\mathcal{G}(\mathbf{r}\tau, \mathbf{r}'\tau')} = \int [dV(\mathbf{r})] P[V(\mathbf{r})] \mathcal{G}(\mathbf{r}\tau, \mathbf{r}'\tau'). \quad (12)$$

explain why the result of 1(d) implies that $\overline{\mathcal{G}(\mathbf{r}\tau, \mathbf{r}'\tau')}$ can be represented by all distinct diagrams of the type

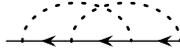


where the thick arrows stands for the disorder averaged Green function and $1 \dots \dots \dots 2 = \gamma\delta(\mathbf{r}_1 - \mathbf{r}_2)$. Explain why this is an expansion in powers of γ , i.e., in the strength of the random potential.

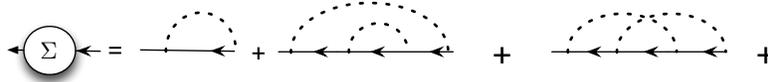
(b) An important concept of diagrammatic approach is the self-energy. Consider a general diagram contributing to the average Green function. This diagram either falls apart into separate pieces when cutting an internal Green function line, or not. Diagrams of the first type are called single-particle reducible, e.g.,



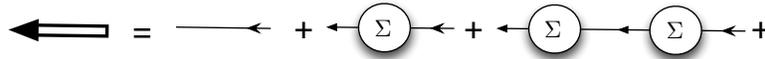
diagrams of the second type single-particle irreducible, e.g.,



Now, consider the sum of all single-particle irreducible diagrams



This quantity, with amputated external Green function legs, is the self-energy. Explain why



and thus



As an equation in matrix notation this becomes

$$\bar{\mathcal{G}} = \mathcal{G}_0 - \mathcal{G}_0 \Sigma \bar{\mathcal{G}}. \quad (13)$$

Show that this can be solved as

$$\bar{\mathcal{G}} = (\mathcal{G}_0^{-1} + \Sigma)^{-1}. \quad (14)$$

(c) Retain only the lowest-order diagram for the self energy of the disorder-averaged Green function and show that

$$\bar{\mathcal{G}}(\mathbf{k}, \omega) = \frac{1}{i\omega_n - (\epsilon_k - \mu) + \frac{i}{2\tau} \text{sgn}\omega_n}. \quad (15)$$

Given an explicit expression for the elastic scattering rate $1/\tau$ and interpret it also in terms of Fermi's Golden Rule.

Problem 3: Feynman diagrams for bosons in a disorder potential (25 points)

While we had fermion systems such as the electron gas in mind in the previous problems, they really treated noninteracting particles. In this case, the particle statistics does not really play any role in determining the nature of the spectrum or the wave functions. So the diagrammatic perturbation theory should also be derivable from the bosonic functional integral. This is the goal of this problem.

Specifically then, rederive the Feynman diagram expansion of problems 1 and 2 starting with the bosonic coherent state functional integral. The basic idea is to use an appropriate expression for the Green function and to expand the exponential of the action in powers of the potential V and identify the resulting terms. (It should be clear that this provides a rather general procedure to develop diagrammatic perturbation expansions, even for more complicated interactions in the Hamiltonian.)