Problem Set 4
Theoretical Solid State Physics (Summer 2021)

Due: Thursday May 14, 2021, before the beginning of class

Problem 1: Fermionic Bogoliubov transformation (10+10+10 points)

This problem extends the Bogoliubiv transformation to fermionic systems with pairing terms. Our starting point is a fermion system with Hamiltonian

$$H = \sum_{k>0} \psi_k^\dagger K_k \psi_k,$$

where we introduced $\psi_k = [c_k, c_{-k}^\dagger]^T$ and

$$K_{\mathbf{k}} = \begin{pmatrix} \xi_k & \delta_k^* \\ \delta_k & -\xi_k \end{pmatrix}$$

with $\delta = |\delta| e^{i\varphi}$.

(a) Show that a transformation $\psi_k = T\psi_k'$ to new operators ψ_k' is canonical if $\mathbf{1} = TT^\dagger$, i.e, when T is unitary. Show how this can be used to solve the Hamiltonian and that the excitation spectrum is fermionic with excitation energies

$$E_{\mathbf{k}} = \sqrt{\xi_k^2 + |\delta_k|^2}.$$

(b) Parametrize

$$T = \begin{pmatrix} u_k^* & -v_k \\ v_k^* & u_k \end{pmatrix}$$

with $u_k = \cos\theta_k e^{i\varphi/2}$ and $v_k = \sin\theta_k e^{i-\varphi/2}$. Compute u_k as well as v_k explicitly. Express the new fermionic operators $\psi_k' = [\gamma_k, \gamma_{-k}^\dagger]^T$ in terms of the c_k and the $c_k^\dagger$.

(c) Show that the ground state can be written as

$$|\mathrm{gs}\rangle = c_{k=0}^\dagger \prod_{k>0} (u_k + v_k c_{-k}^\dagger c_k^\dagger)|\mathrm{vac}\rangle.$$

Problem 2: 1D anisotropic XY model in a transverse magnetic field (10 + 10 + 20 points)

Consider the anisotropic 1D spin-$\frac{1}{2}$ XY model in a transverse magnetic field B,

$$H = -\sum_j \left[(J+\Delta) S_j^x S_{j+1}^x + (J-\Delta) S_j^y S_{j+1}^y + B S_j^z \right].$$

In the limit $\Delta = 0$, this is the XY model in a transverse field.

(a) Use the Jordan-Wigner transformation to rewrite this Hamiltonian as a quadratic fermion problem.

(b) Diagonalize the fermion Hamiltonian by Fourier transforming to momentum space (lattice of M sites with periodic boundary conditions),

$$f_j = \frac{1}{\sqrt{M}} \sum_k c_k e^{ikj},$$

and using the fermionic Bogoliubov transformation (see problem 1). You should find the excitation spectrum

$$E_k = \sqrt{(-J\cos k - B)^2 + (\Delta \sin k)^2}$$

(c) Show that the excitation spectrum becomes gapless for $\Delta = 0$ and for $|J| = |B|$. These points correspond to phase transitions. Draw a phase diagram for fixed J as a function of B and Δ and identify the various phases.

Problem 3: Duality transformation for the transverse field Ising model (20+20 points)

The model in Problem 2 with $\Delta = J$ is known as the transverse field Ising model (also: quantum Ising model). Traditionally, one writes the exchange coupling in terms of S^z and applies the transverse field in the x direction,

$$H = -J \sum_j \left[\sigma_j^z \sigma_{j+1}^z + g\sigma_j^x\right].$$

Here, we also write the model in terms of Pauli matrices (i.e., dropping the factors of $1/2$ in the spin operators), assume ferromagnetic coupling $J > 0$, and write the transverse field as Jg.

(a) Define new operators through

$$S_i^z = \prod_{j \leq i} \sigma_j^x \quad ; \quad S_i^x = \sigma_i^z \sigma_{i+1}^z.$$

You can think of these operators as living on the bonds of the original lattice. First explain why these are domain-wall operators, i.e., that S_i^x detects the existence of a domain wall at the bond $(i, i+1)$ and that S_i^z can be thought of a domain-wall creation operator in the ferromagnetic phase.

(b) Show that these new operators are also Pauli operators, i.e., they square to unity, anticommute on the same site, and commute on different sites. (The angular-momentum algebra is automatically obeyed when defining $S_i^y = -iS_i^z S_i^x$.)

(c) Invert the duality transformation to obtain

$$\sigma_i^z = \prod_{j<i} S_j^x \quad ; \quad \sigma_i^x = S_i^z S_{i-1}^z.$$

and write the Hamiltonian in terms of the new Pauli operators. Show that this maps the tranverse field Ising model onto itself, with $J \to Jg$ and $g \to 1/g$, i.e., the Ising model is self dual under this transformation. What happens at the fixed point $g = 1$?

It is worthwhile to think about this further. The duality transformation obviously maps strong to weak coupling and vice versa, i.e., it maps the ferromagnetic and paramagnetic phases onto one another. In the original model, the ferromagnetic phase has an order parameter σ_i^z, which has a nonzero expectation value in the symmetry-broken ground state. Thus, the duality transformation implies that $S_i^z = \prod_{j \leq i} \sigma_j^x$ has a finite expectation value in the paramagnetic phase and may thus be referred to as a disorder parameter (nonzero in the disordered phase and zero in the ordered phase). Since $S_i^z = \prod_{j \leq i} \sigma_j^x$ is a domain-wall creation operator, the paramagnetic phase can be thought of as a condensate of domain walls. (Remember that the Bose field has a finite expectation value in the Bose-condensed phase.)

This duality transformation can also be extended to the transverse field Ising model in 2D. In a 2D square lattice, the number of bonds is twice the number of sites. Thus, different configurations of the bond spins must correspond to the same configuration of the original spins. For this reason, the dual of the 2D transverse field Ising model is actually a $\mathbb{Z}_2$ lattice gauge theory. (Remember the redundance in describing the physical field configurations in electromagnetism, the most familiar gauge theory, when using the electromagnetic potentials.)