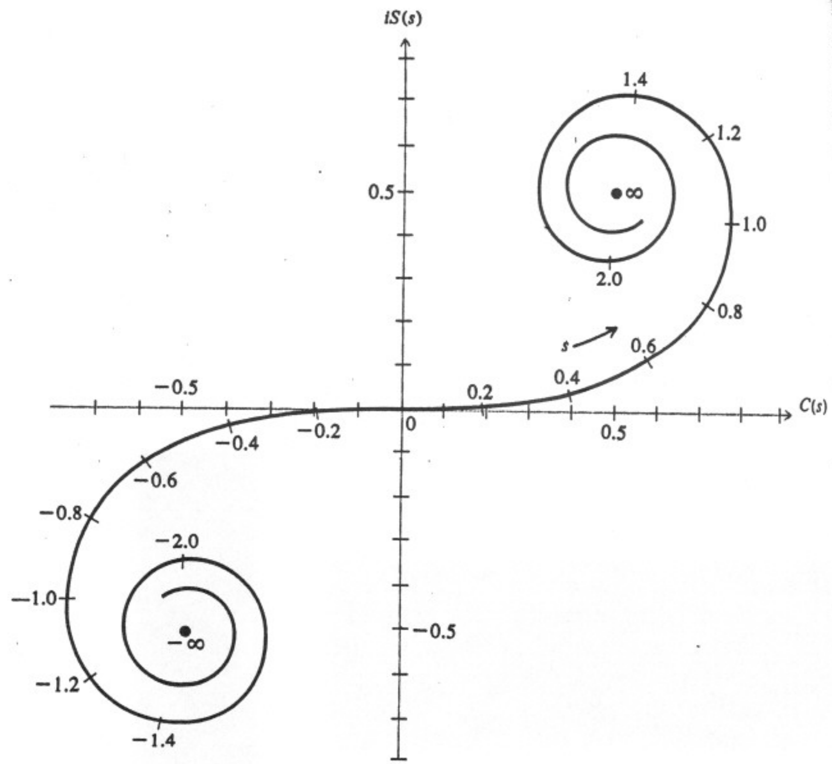


Figure 5.24. (a) The Cornu spiral. The scale of w is marked on the curve.

spiral this would be U_1 times the length of the line from $-\infty$ to ∞ [Figure 5.24(b)]. Setting this equal to U_0, we can express the general case in the normalized form

$$U_p = \frac{U_0}{(1+i)^2}\,[C(u) + iS(u)]_{u_1}^{u_2}\,[C(v) + iS(v)]_{v_1}^{v_2} \tag{5.47}$$

Strictly speaking, very large values of the parameters u, v, or s would be inconsistent with the approximation expressed by Equation (5.41). However, in normal cases of interest most of the contribution to U_p comes from the lower-order Fresnel zones in the aperture, corresponding to low values of the above parameters, hence the approximation is still valid.

Slit and Straightedge Fresnel diffraction by a long slit is treated as a limiting case of a rectangular aperture, namely, by letting $u_1 = -\infty$ and $u_2 = +\infty$ in Equation (5.47). This yields the formula

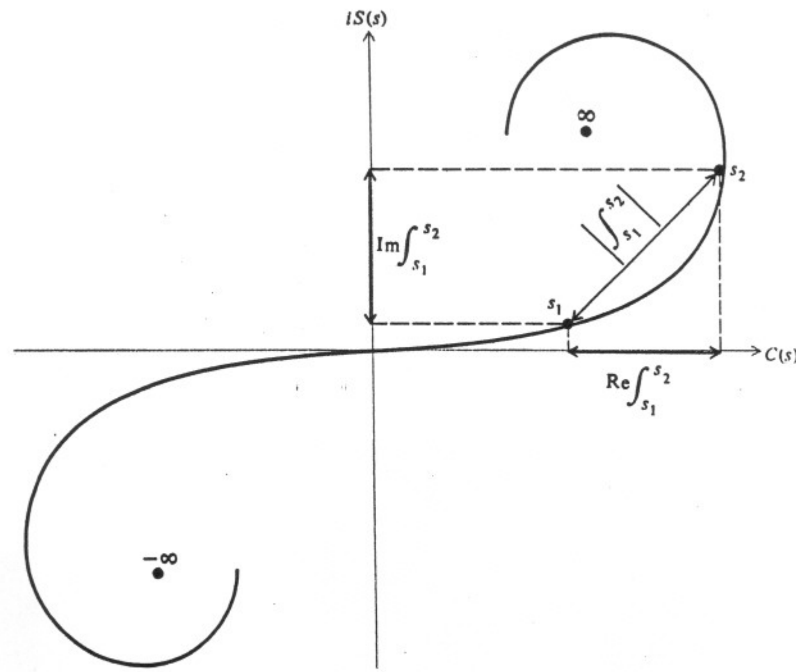


Figure 5.24. (b) Evaluation of Fresnel integrals with the Cornu spiral.

$$U_p = \frac{U_0}{1+i}\,[C(v) + iS(v)]_{v_1}^{v_2} \tag{5.48}$$

for the slit where v_1 and v_2 define the slit edges.

The straightedge is similarly taken as a limiting case of a slit: $v_1 = -\infty$. This gives

$$\begin{aligned} U_p &= \frac{U_0}{1+i}\,[C(v) + iS(v)]_{-\infty}^{v_2} \\ &= \frac{U_0}{1+i}\left[C(v_2) + iS(v_2) + \frac{1}{2} + \frac{1}{2}\,i\right] \end{aligned} \tag{5.49}$$

which is a function of only the one variable v_2. This variable specifies the position of the diffracting edge. If the receiving point P is exactly at the geometrical shadow edge, then $v_2 = 0$. We have then $U_p = [U_0/(1+i)]\,(\frac{1}{2} + \frac{1}{2}i) = \frac{1}{2}U_0$. Hence the amplitude at the shadow edge is one half, and the irradiance is one fourth the unobstructed value. A plot of $I_p = |U_p|^2$ as given by Equation (5.49) is shown in Figure 5.25. Here I_p is plotted as a function of v_2. This is equivalent to having a fixed position for the receiving point and varying the posi-

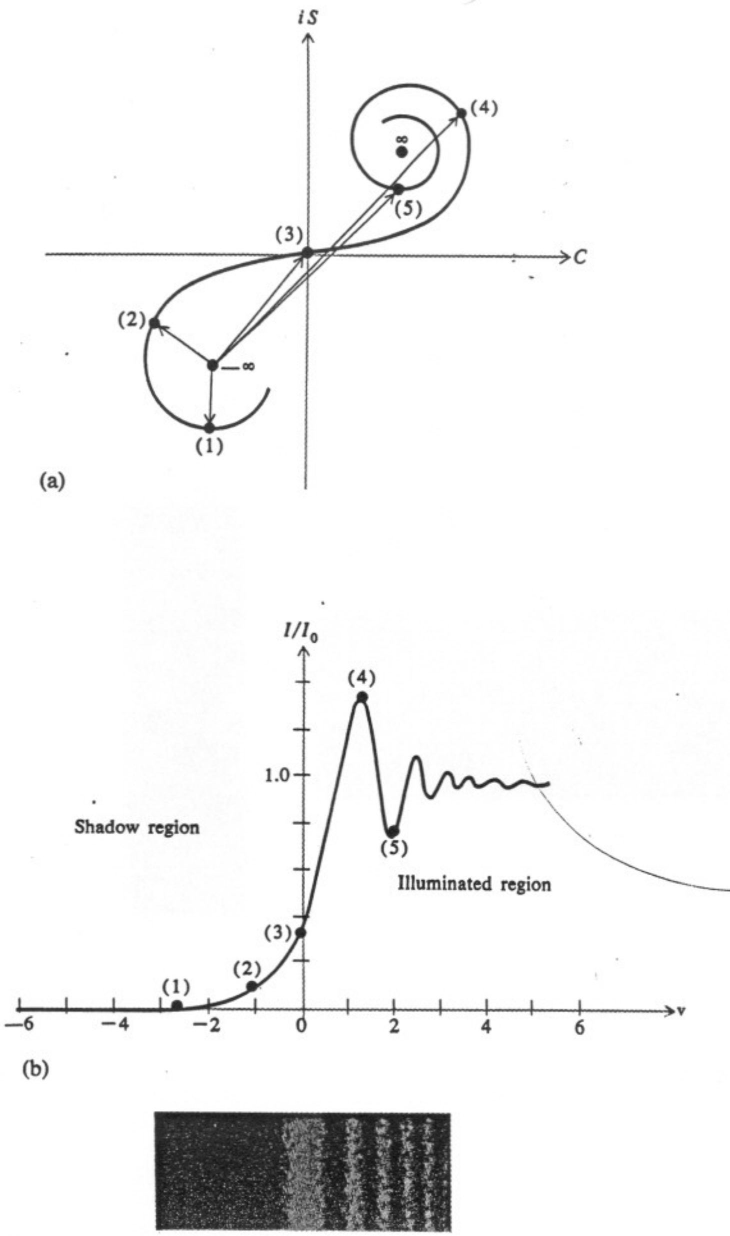


Figure 5.25. Fresnel diffraction by a straightedge. (a) Points on the Cornu spiral; (b) corresponding points on the intensity curve; $v = 0$ defines the geometrical shadow edge. A photograph of the diffraction pattern is shown below.

tion of the diffracting edge. The result is virtually the same as a diffraction pattern. From the graph it can be seen that the irradiance falls off rapidly and monotonically in the shadow zone ($v_2 < 0$) as $v_2 \to -\infty$. On the other hand, in the illuminated zone ($v_2 > 0$) the irradiance oscillates with diminishing amplitude about the unobstructed value U_0 as $v_2 \to +\infty$. The highest irradiance occurs just inside the illuminated region at the point $v_2 \approx 1.25$, where I_p is 1.37 times the irradiance of the unobstructed wave. This is seen as a bright fringe next to the geometrical shadow.

5.6 Applications of the Fourier Transform to Diffraction

Let us return to the discussion of Fraunhofer diffraction. We now consider the general problem of diffraction by an aperture having not only an arbitrary shape, but also an arbitrary transmission including phase retardation, which may vary over different parts of the aperture.

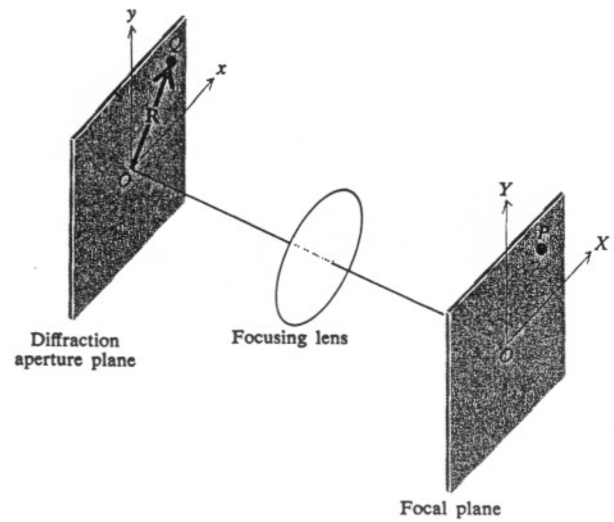


Figure 5.26. Geometry of the general diffraction problem.

We choose coordinates as indicated in Figure 5.26. The diffracting aperture lies in the xy plane, and the diffraction pattern appears in the XY plane, which is the focal plane of the focusing lens. According to elementary geometrical optics, all rays leaving the diffracting aperture in a given direction, specified by direction cosines α, β, and γ, are brought to a common focus. This focus is located at the point $P(X,Y)$ where $X \approx L\alpha$ and $Y \approx L\beta$, L being the focal length of the lens. The assumption is made here that α and β are small, so that $\alpha \approx \tan\alpha$ and $\beta \approx \tan\beta$. We also assume that $\gamma \approx 1$.

Now the path difference δr, between a ray starting from the point $Q(x,y)$ and a parallel ray starting from the origin O, is given by $\mathbf{R}\cdot\hat{\mathbf{n}}$, (Figure 5.27), where $\mathbf{R} = \hat{\mathbf{i}}x + \hat{\mathbf{j}}y$ and $\hat{\mathbf{n}}$ is a unit vector in the direction

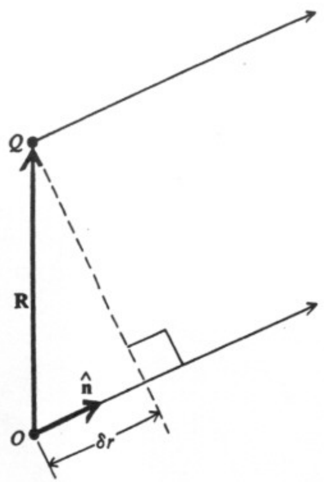


Figure 5.27. Path difference between two parallel rays of light originating from points O and Q in the xy plane.

of the ray. Since $\hat{\mathbf{n}}$ can be expressed as $\hat{\mathbf{n}} = \hat{\mathbf{i}}\alpha + \hat{\mathbf{j}}\beta + \hat{\mathbf{k}}\gamma$, then

$$\delta r = \mathbf{R}\cdot\hat{\mathbf{n}} = x\alpha + y\beta = x\frac{X}{L} + y\frac{Y}{L} \tag{5.50}$$

It follows that the fundamental diffraction integral [Equation (5.16)] giving the diffraction pattern in the XY plane is, aside from a constant multiplying factor, expressible in the form

$$U(X,Y) = \iint e^{ik\delta r}\, d\mathscr{A} = \iint e^{ik(xX+yY)/L}\, dx\, dy \tag{5.51}$$

This is the case for a uniform aperture.

For a uniform rectangular aperture the double integral reduces to the product of two one-dimensional integrals. The result is stated earlier in Section 5.4.

For a nonuniform aperture we introduce a function $g(x,y)$ called the *aperture function*. This function is defined such that $g(x,y)\,dx\,dy$

is the amplitude of the diffracted wave originating from the element of area $dx\,dy$. Thus instead of Equation (5.51), we have the more general formula

$$U(X,Y) = \iint g(x,y)\, e^{i(Xx+Yy)}\, dx\, dy \tag{5.52}$$

It is convenient at this point to introduce the quantities

$$\mu = \frac{kX}{L} \quad \text{and} \quad \nu = \frac{kY}{L} \tag{5.53}$$

μ and ν are called *spatial frequencies*, although they have the dimensions of reciprocal length, that is, wavenumber. We now write Equation (5.52) as

$$U(\mu,\nu) = \iint g(x,y)\, e^{i(\mu x+\nu y)}\, dx\, dy \tag{5.54}$$

We see that the functions $U(\mu,\nu)$ and $g(x,y)$ constitute a two-dimensional Fourier transform pair. The diffraction pattern, in this context, is actually a Fourier resolution of the aperture function.

Consider as an example a grating. For simplicity we treat it as a one-dimensional problem. The aperture function $g(y)$ is then a periodic step function as shown in Figure 5.28. It is represented by a Fourier series of the form

$$g(y) = g_0 + g_1\cos(\nu_0 y) + g_2\cos(2\nu_0 y) + \cdots \tag{5.55}$$

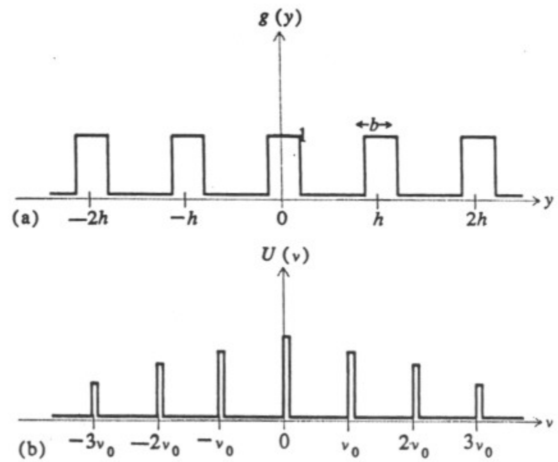


Figure 5.28. Aperture function for a grating and its Fourier transform.

The fundamental spatial frequency ν_0 is given by the periodicity of the grating, namely,

$$\nu_0 = \frac{2\pi}{h} \tag{5.56}$$

where h is the grating spacing. This dominant spatial frequency appears in the diffraction pattern as the first-order maximum, the amplitude of which is proportional to g_1. Maxima of higher order correspond to higher Fourier components of the aperture function $g(y)$. Thus if the aperture function were of the form of a cosine function $g_0 + g_1 \cos(\nu_0 y)$ instead of a periodic step function, then the diffraction pattern would consist only of the central maximum and the two first-order maxima. Second or higher diffraction orders would not appear.

Apodization Apodization (literally "to remove the feet") is the name given to any process by which the aperture function is altered in such a way as to produce a redistribution of energy in the diffraction pattern. Apodization is usually employed to reduce the intensity of the secondary diffraction maxima.

It is perhaps easiest to explain the theory of apodization by means of a specific example. Let the aperture consist of a single slit. The aperture function in this case is a single step function: $g(y) = 1$ for $-b/2 < y < b/2$ and $g(y) = 0$ otherwise (Figure 5.29). The corresponding diffraction pattern, expressed in terms of spatial frequencies, is

$$U(\nu) = \int_{-b/2}^{+b/2} e^{i\nu y}\, dy = b\, \frac{\sin(\frac{1}{2}\nu b)}{(\frac{1}{2}\nu b)} \tag{5.57}$$

This is equivalent to the normal case already discussed in Section 5.5.

Suppose now that the aperture function is altered by apodizing in such a way that the resultant aperture transmission is a cosine function: $g(y) = \cos(\pi y/b)$ for $-b/2 < y < b/2$ and zero otherwise, as shown in Figure 5.29. This could be accomplished, for example, by means of a suitably coated-glass plate placed over the aperture. The new diffraction pattern is given by

$$\begin{aligned} U(\nu) &= \int_{-b/2}^{+b/2} \cos\left(\frac{\pi y}{b}\right) e^{i\nu y}\, dy \\ &= \cos(\nu b/2) \left(\frac{1}{\nu - \pi/b} - \frac{1}{\nu + \pi/b}\right) \end{aligned} \tag{5.58}$$

A comparison of the two diffraction patterns is shown graphically in the figure. The result of apodization in this case is a substantial reduction in the secondary maxima relative to the central maximum; in other words, apodization has suppressed the higher spatial frequencies.

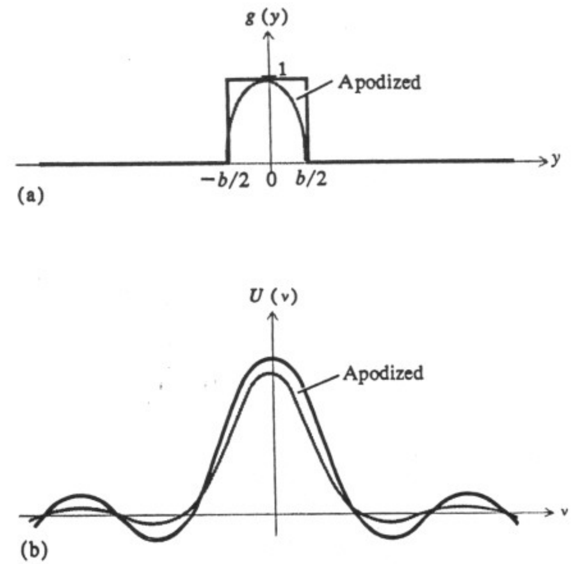


Figure 5.29. (a) Aperture functions for a slit and an apodized slit; (b) the Fourier transforms.

In a similar way it is possible to apodize the circular aperture of a telescope so as to reduce greatly the relative intensities of the diffraction rings that appear around the images of stars (discussed in Section 5.5). This enhances the ability of the telescope to resolve the image of a dim star near that of a bright one.

Spatial Filtering Consider the diagram shown in Figure 5.30. Here the xy plane represents the location of some *coherently* illuminated object.[3] This object is imaged by an optical system (not shown), the image appearing in the $x'y'$ plane. The diffraction pattern $U(\mu,\nu)$ of the object function $g(x,y)$ appears in the $\mu\nu$ plane. This plane is analogous to the XY plane in Figure 5.26. Hence, from Equation (5.54) $U(\mu,\nu)$ is the Fourier transform of $g(x,y)$. The image function $g'(x',y')$ that appears in the $x'y'$ plane is, in turn, the Fourier transform of $U(\mu,\nu)$. Now if *all* spatial frequencies in the range $\mu = \pm\infty$, $\nu = \pm\infty$ were transmitted equally by the optical system, then, from the properties of the Fourier transform, the image function $g'(x'y')$

[3] For a discussion of the theory of spatial filtering with incoherent illumination see Reference [10].

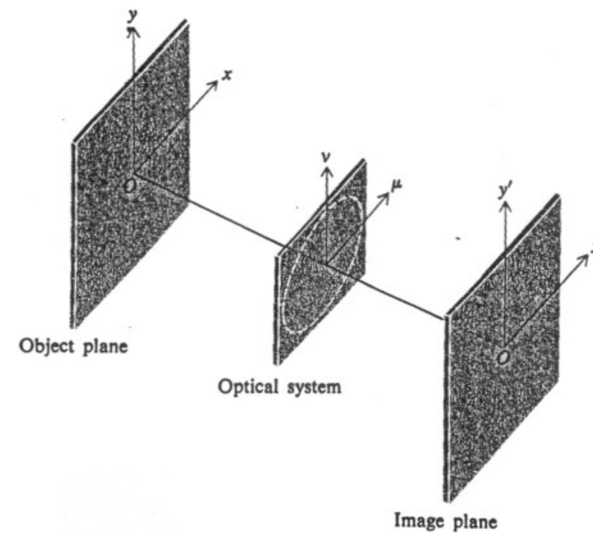


Figure 5.30. Geometry for the general problem of image formation by an optical system.

would be *exactly* proportional to the object function $g(x,y)$; that is, the image would be a true reproduction of the object. However, the finite size of the aperture at the $\mu\nu$ plane limits the spatial frequencies that are transmitted by the optical system. Furthermore there may be lens defects, aberrations, and so forth, which result in a modification of the function $U(\mu,\nu)$. All of these effects can be incorporated into one function $T(\mu,\nu)$ called the *transfer function* of the optical system. This function is defined implicitly by the equation

$$U'(\mu,\nu) = T(\mu,\nu)\, U(\mu,\nu)$$

Thus

$$g'(x', y') = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} T(\mu,\nu)U(\mu,\nu)\, e^{-i(\mu x' + \nu y')}\, d\mu\, d\nu \qquad (5.59)$$

that is, the image function is the Fourier transform of the product $T(\mu,\nu)\cdot U(\mu,\nu)$. The limits of integration are $\pm\infty$ in a formal sense only. The actual limits are given by the particular form of the transfer function $T(\mu,\nu)$.

The transfer function can be modified by placing various screens and apertures in the $\mu\nu$ plane. This is known as *spatial filtering*. The situation is quite analogous to the filtering of an electrical signal by means of a passive electrical network. The object function is the input signal, and the image function is the output signal. The optical system

acts like a filter that allows certain spatial frequencies to be transmitted but rejects others.

Suppose, for example, that the object is a grating so that the object function is a periodic step function. This case can be treated as a one-dimensional problem. The object function $f(y)$ and its Fourier transform $U(\nu)$ are then just those shown in Figure 5.28. Now let the aperture in the $\mu\nu$ plane be such that only those spatial frequencies that lie between $-\nu_{max}$ and $+\nu_{max}$ are transmitted. This means that we have low-pass filtering. From Equation (5.53) we have $\nu_{max} = kb/f$, where $2b$ is the physical width of the aperture in the $\mu\nu$ plane. The transfer function for this case is a step function: $T(\nu) = 1$, $-\nu_{max} < \nu < +\nu_{max}$, and zero otherwise. The image function is, accordingly,

$$g'(y') = \int_{-\nu_{max}}^{+\nu_{max}} U(\nu) e^{-i\nu y'}\, d\nu \qquad (5.60)$$

Without going into the details of the calculation of $g'(y')$, we show in Figure 5.31(a) a graphical plot for some arbitrary choice of ν_{max}. Instead of the sharp step function that constitutes the object, the image is rounded at the corners and also shows small periodic variations.

A high-pass optical filter is obtained by placing in the $\mu\nu$ plane a screen that blocks off the central part of the diffraction pattern. This part of the diffraction pattern corresponds to the low frequencies. The approximate form of the resulting image function is shown in Figure 5.31(b). Only the edges of the grating steps are now visible in the image plane. *The edge detail comes from the higher spatial frequencies.*

A practical example of spatial filtering is the *pinhole spatial filter* which is used in laser work to reduce the spurious fringe pattern that always occurs in the output beam of a helium–neon laser. The beam is brought to a sharp focus by means of a short-focal-length lens. A fine pinhole placed at the focal point constitutes the filter, which removes the higher spatial frequencies and hence improves the beam quality of the laser output. A second lens can be used to render the beam parallel.

Phase Contrast and Phase Gratings The method of phase contrast was invented by the Dutch physicist Zernike. It is used to render visible a transparent object whose index of refraction differs slightly from that of a surrounding transparent medium. Phase contrast is particularly useful in microscopy for examination of living organisms, and so forth. In essence, the method consists of the use of a special type of spatial filter.

To simplify the theory of phase contrast, we shall treat the case of a so-called "phase grating" consisting of alternate strips of high- and

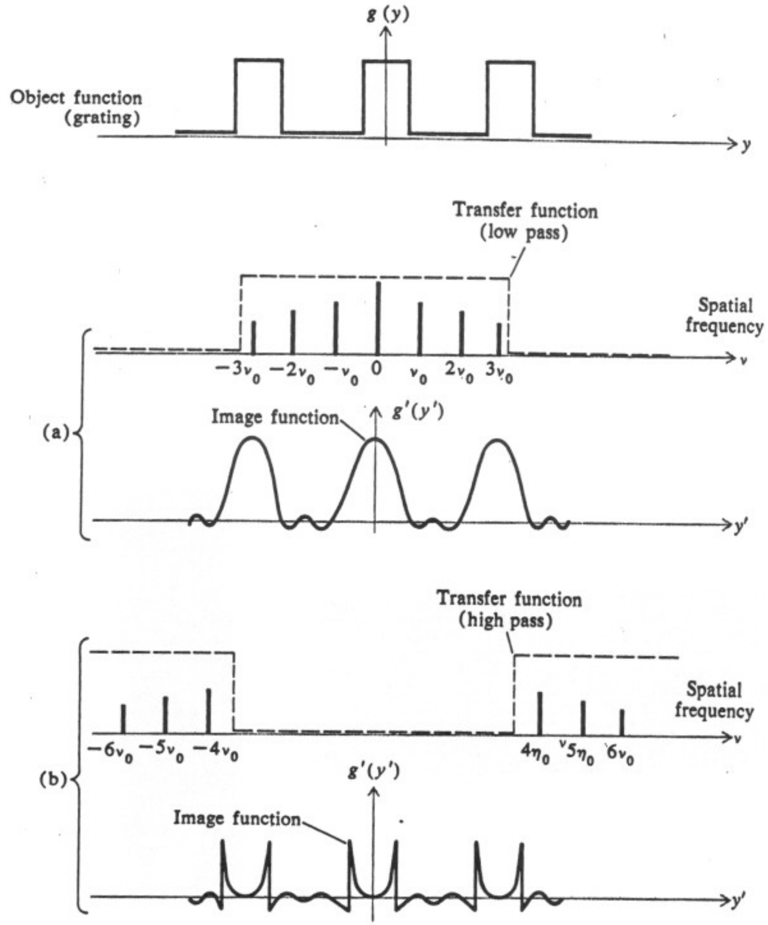


Figure 5.31. Graphs illustrating spatial filtering. (a) Low-pass filtering; (b) high-pass filtering.

low-index material, all strips being perfectly transparent. The grating is coherently illuminated and constitutes the object. The object function is thus represented by the exponential

$$g(y) = e^{i\phi(y)} \tag{5.61}$$

where the phase factor $\phi(y)$ is a periodic step function as shown in Figure 5.32(a). The "height" of the step is the optical-phase difference between the two kinds of strips; that is, $\Delta\phi = kz\,\Delta n$, where z is the thickness and Δn is the difference between the two indices of

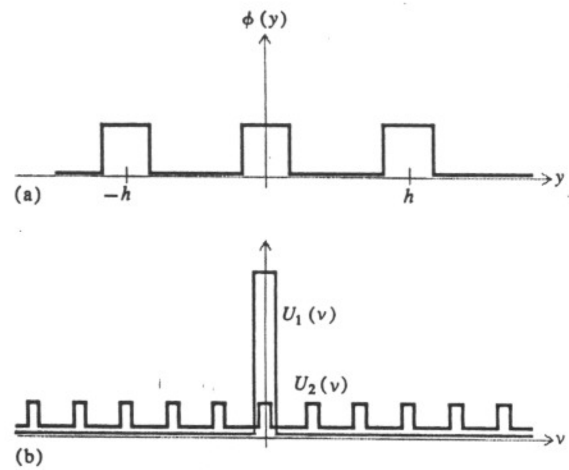


Figure 5.32. (a) The phase function of a periodic phase grating; (b) Fourier transforms of the aperture U_1 and the grating U_2.

refraction. If we assume that this phase difference is very small, then to a good approximation, we can write

$$g(y) = 1 + i\phi(y) \tag{5.62}$$

The Fourier transform of the above function is

$$U(\nu) = \int_{-\infty}^{\infty} [1 + i\phi(y)]e^{i\nu y}\,dy = \int_{-b/2}^{+b/2} e^{i\nu y}\,dy + i\int_{-b/2}^{+b/2} \phi(y)\,e^{i\nu y}\,dy$$

$$= U_1(\nu) + iU_2(\nu) \tag{5.63}$$

Here $U_1(\nu)$ represents the diffraction pattern of the whole-object aperture. It is essentially zero everywhere except for $\nu \approx 0$; that is, $U_1(\nu)$ contains only very low spatial frequencies. On the other hand, $U_2(\nu)$ represents the diffraction pattern of the periodic step function $\phi(y)$. The two functions are plotted in Figure 5.32(b).

By virtue of the factor i in the result, $U_1 + iU_2$, the two components U_1 and iU_2 are 90 degrees out of phase. The essential trick in the phase-contrast method consists of inserting a spatial filter in the $\mu\nu$ plane, which has the property of shifting the phase of iU_2 by an additional 90 degrees. In practice this is accomplished by means of a device known as a *phase plate*. The physical arrangement is shown in Figure 5.33. The phase plate is just a transparent-glass plate having a small section whose optical thickness is $\frac{1}{4}$ wavelength greater than the remainder of the plate. This thicker section is located in the central part of the $\mu\nu$ plane, that is, in the region of low spatial frequencies. The result of inserting the phase plate is to change the function

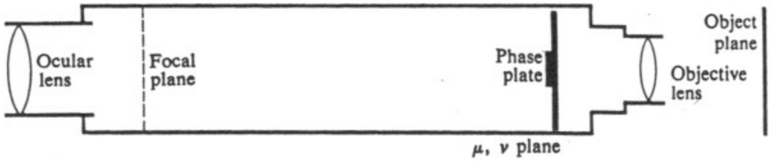


Figure 5.33. Physical arrangement of the optical elements for phase contrast microscopy.

$U_1 + iU_2$ to $U_1 + U_2$. The new image function is given by the Fourier transform of the new $U(\nu)$, namely,

$$\begin{aligned} g'(y') &= \int U_1(\nu)e^{-i\nu y'}\,d\nu + \int U_2(\nu)e^{-i\nu y'}\,d\nu \\ &= g_1(y') + g_2(y') \end{aligned} \tag{5.64}$$

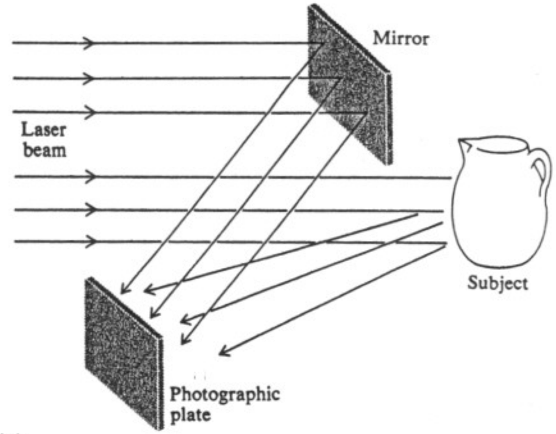
Now the first function g_1 is just the image function of the whole-object aperture. It represents the constant background. The second function g_2 is the image function for a regular grating of alternate transparent and opaque strips. This means that the phase grating has been rendered visible. It appears in the image plane as alternate bright and dark strips. Although the above analysis has been for a periodic grating, a similar argument can be applied to a transparent-phase object of any shape.

The method of optical-phase contrast has a close analogy in electrical communications. A phase-modulated signal is converted into an amplitude-modulated signal by introduction of a phase shift of 90 degrees to the carrier frequency. This is essentially what the phase plate does in the phase-contrast method. The net result is that phase modulation in the object is converted into amplitude modulation in the image.

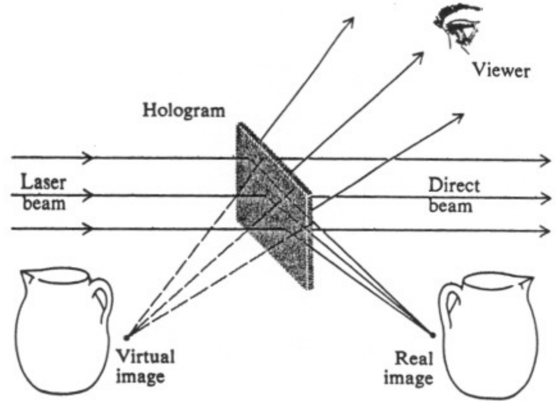
5.7 Reconstruction of the Wave Front by Diffraction. Holography

An unusual and interesting method of producing an image—known as the method of *wave-front reconstruction*—has recently become of importance in the field of optics. Although the basic idea was originally proposed by Gabor in 1947 [12], it attracted little attention until the highly coherent light of the laser became available.

In this method a special diffraction screen, called a *hologram*, is used to reconstruct in detail the wave field emitted by the subject. To make the hologram the output from a laser is separated into two beams, one of which illuminates the subject. The other beam, called the *reference beam*, is reflected onto a fine-grained photographic film



(a)



(b)

Figure 5.34. (a) Arrangement for producing a hologram; (b) use of the hologram in producing the real and virtual images.

by means of a mirror. The film is exposed simultaneously to the reference beam and the reflected laser light from the subject [Figure 5.34(a)]. The resulting complicated interference pattern recorded by the film constitutes the hologram. It contains all the information needed to reproduce the wave field of the subject.

In use the developed hologram is illuminated with a single beam from a laser as shown in Figure 5.34(b). Part of the resulting diffracted wave field is a precise, three-dimensional copy of the original wave reflected by the subject. The viewer looking at the hologram sees the image in depth and by moving his head can change his perspective of the view.

In order to simplify the discussion of the theory of holography, we shall assume that the reference beam is collimated; that is, it consists of plane waves, although this is not actually necessary in practice. Let x and y be the coordinates in the plane of the recording photographic plate, and let $U(x,y)$ denote the complex amplitude of the reflected wave front in the xy plane. Since $U(x,y)$ is a complex number, we can write it as

$$U(x,y) = a(x,y)e^{i\phi(x,y)} \tag{5.65}$$

where $a(x,y)$ is real.

Similarly, let $U_0(x,y)$ denote the complex amplitude of the reference beam. Since this beam is plane, we can write

$$U_0(x,y) = a_0\, e^{i(\mu x + \nu y)} \tag{5.66}$$

where a_0 is a constant and μ and ν are the spatial frequencies of the reference beam in the xy plane. They are given by

$$\mu = k \sin \alpha \qquad \nu = k \sin \beta \tag{5.67}$$

in which k is the wave number of the laser light, and α and β specify the direction of the reference beam.

The irradiance $I(x,y)$ that is recorded by the photographic film is thus given by the expression

$$\begin{aligned} I(x,y) &= \|U + U_0\|^2 = a^2 + a_0^2 + aa_0e^{i[\phi(x,y)-\mu x-\nu y]} + aa_0e^{-i[\phi(x,y)-\mu x-\nu y]} \\ &= a^2 + a_0^2 + 2aa_0 \cos\,[\phi(x,y) - \mu x - \nu y] \end{aligned} \tag{5.68}$$

This is actually an interference pattern. It contains information in the form of amplitude and phase modulations of the spatial frequencies of the reference beam. The situation is somewhat analogous to the impression of information on the carrier wave of a radio transmitter by means of amplitude or phase modulation.

When the developed hologram is illuminated with a single beam U_0 similar to the reference beam, the resulting transmitted wave U_T is proportional to U_0 times the transmittance of the hologram at the point (x,y). The transmittance will be proportional to $I(x,y)$. Hence, except for a constant proportionality factor that we ignore,

$$\begin{aligned} U_T(x,y) &= U_0I = a_0(a^2 + a_0^2)e^{i(\mu x+\nu y)} + a_0^2a\ e^{i\phi} + a_0^2ae^{-i(\phi-2\mu x-2\nu y)} \\ &= (a^2 + a_0^2)\,U_0 + a_0^2U + a^2U^{-1}U_0^{-2} \end{aligned} \tag{5.69}$$

The hologram acts somewhat like a diffraction grating. It produces a direct beam and two first-order diffracted beams on either side of the direct beam [Figure 5.34(b)]. The term $(a^2 + a_0^2)U_0$ in Equation (5.69) comprises the direct beam. The term a_0^2U represents one of the diffracted beams. Since it is equal to a constant times U, this beam is the one that reproduces the reflected light from the subject and forms the virtual image. The last term represents the other diffracted beam and gives rise to a real image.

We shall not attempt to prove the above statements in detail. They can be verified by considering a very simple case, namely, that in which the subject is a single white line on a dark background. In this case the hologram turns out to be, in fact, a simple periodic grating. The zero order of the diffracted light is the direct beam, whereas the two first orders on either side comprise the virtual and the real images.

In holography the viewer always sees a positive image whether a positive or a negative photographic transparency is used for the hologram. The reason for this is that a negative hologram merely produces a wave field that is shifted 180 degrees in phase with respect to that of a positive hologram. Since the eye is insensitive to this phase difference, the view seen by the observer is identical in the two cases.

Remarkable technical advances have been made in the field of holography in recent years. Holography in full color is possible by using three different laser wavelengths instead of just one, the holographic record being on black-and-white film. The holographic principle has been extended to include the use of acoustic waves for imaging in optically opaque media and to microwaves for long-distance holography.

Holographic Interferometry One of the most notable applications of holography is in the field of interferometry. In this application the surface to be tested can be irregular and diffusely reflecting instead of smooth and highly polished as is required for the ordinary Michelson and Twyman-Green type of interferometric work. In *double-exposure* holographic interferometry two separate exposures are made on a single recording film. If the surface under study undergoes any deformation or movement during the time interval between exposures, such movement is revealed on the reconstructed image in the form of interference fringes. In *double-pulse* holography the two exposures are produced by short, intense laser pulses from a high-power pulsed laser. These pulses are closely spaced in time so that the holographic image fringes can show motion, vibration patterns, and so on. The method is especially useful for nondestructive testing. For more information on the subject of holography, the reader is encouraged to consult a text such as *An Introduction to Coherent Optics and Holography* by G. W. Stroke [38].