APPENDIX I

Relativistic Optics

I.1 The Michelson-Morley Experiment

The famous Michelson-Morley experiment, performed in 1887, was designed to measure the absolute velocity of the earth's motion in space by means of light waves.

A diagram of the optical arrangement is shown in Figure I.1. The

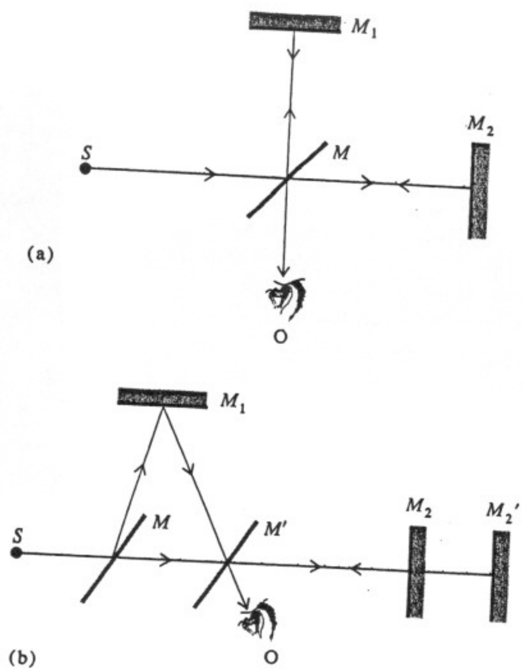

Figure I.1. Simplified diagram of the Michelson-Morley experiment.

apparatus is essentially an optical interferometer. A beam of light from a source S is split into two beams by a half-silvered mirror M. One beam is reflected to a mirror M_1, which in turn reflects the light directly back to M. The other beam is transmitted directly to the mirror M_2, which also reflects the light back to M. The two partial beams then unite at M, part of the combined light going to an observer O who sees an interference pattern of bright and dark fringes. The interference pattern can be made to shift by one fringe by displacing either of the two mirrors M_1 or M_2 a distance of $\frac{1}{4}$ wavelength.

If mirrors M_1 and M_2 are both located at precisely the same distance from M and if the apparatus does not move during the time that light is reflected back and forth, then the two waves return to M at the same phase so that a bright fringe is seen at O. Suppose, however, that the whole apparatus is moving in the direction of the initial beam SM. The paths of the beams will then be as shown by the directed lines in the figure. The times taken by the two partial waves in their respective journeys are no longer the same if it is assumed that light travels with a constant speed c in some medium. The situation is analogous to the case of two swimmers in a stream, one swimmer going upstream and back, the other going across the stream and returning.

To analyze the situation quantitatively, let us suppose that the speed of the apparatus through the medium is u. Then the wave moving toward M_2 travels with a speed $c - u$ relative to the apparatus. On its return this wave travels with relative speed $c + u$. The total time for the round trip is therefore

$$t_2 = \frac{d}{c-u} + \frac{d}{c+u} = \frac{2cd}{c^2 - u^2}$$

in which d is the distance OM_2. On the other hand, the wave reflected by M_1 travels along the path $MM_1'O$, as shown. If we call t_1 the total time for the round trip in this case, then the distance MM_1' is equal to $\sqrt{d^2 + (\frac{1}{4})u^2 t_1^2}$. Thus

$$t_1 = \left(\frac{2}{c}\right)\sqrt{d^2 + \tfrac{1}{4}u^2 t_1^2}$$

Solving for t_1, we get

$$t_1 = \frac{2d}{\sqrt{c^2 - u^2}}$$

The time difference Δt between the two paths is accordingly

$$\Delta t = t_2 - t_1 = 2d\left(\frac{c}{c^2 - u^2} - \frac{1}{\sqrt{c^2 - u^2}}\right) = \frac{du^2}{c^3} + \cdots$$

This corresponds to a phase difference

$$\Delta\phi = \omega\,\Delta t = \frac{2\pi c}{\lambda}\,\Delta t \approx \frac{2\pi d}{\lambda}\,\frac{u^2}{c^2}$$

where λ is the wavelength of the light.

In their experiment, Michelson and Morley obtained an effective distance d of 10 m by multiple reflections as indicated in Figure I.2.

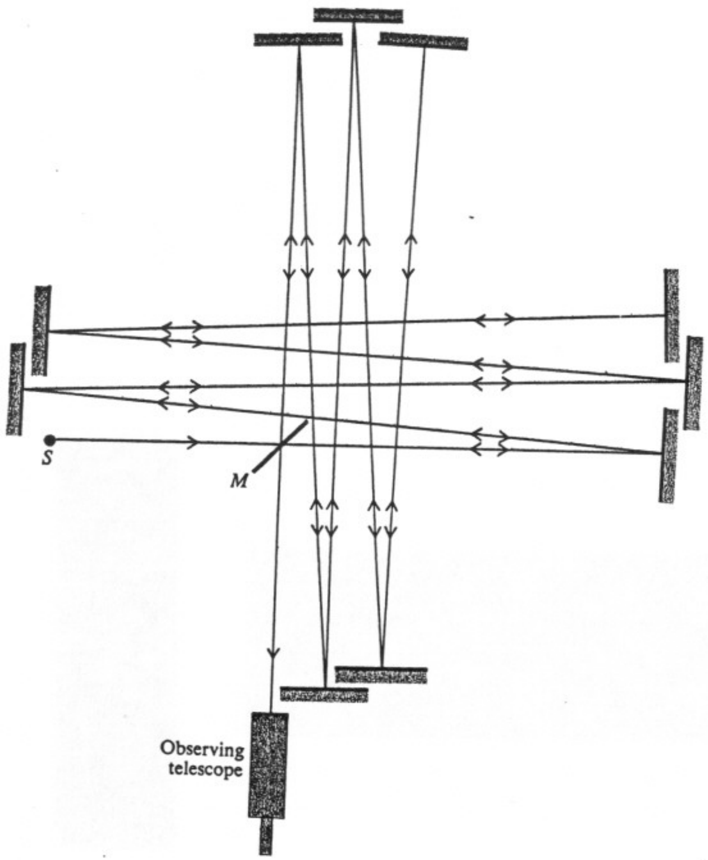

Figure I.2. Actual light path in the Michelson-Morley experiment.

The experiment was performed by floating the entire apparatus in a pool of mercury and observing the fringes as the apparatus was rotated through an angle of 90 degrees. This would cause either of the two beams to be alternately parallel or perpendicular to the earth's motion. In its orbital motion around the sun, the earth's speed is about 10^{-4} c. The expected shift with yellow light, 5900 Å, was about one third of a fringe. *Actually there was no observable shift at all.* This negative result came as a surprise to the scientific world. It was in contradiction to the (then) accepted idea concerning electromagnetic radiation, namely, that such radiation must have a medium for its transmission through space. This medium, called the *ether,* was

supposed to be an all-pervading substance, and numerous calculations concerning its properties had been carried out, including some by Maxwell.

The Michelson-Morley experiment has been repeated many times by different observers with essentially the same negative results. There have been some reports of measurable fringe shifts, but none anywhere near as large as should be predicted by the orbital speed of the earth. This is actually a minimum speed since the speed of the whole solar system, due to rotation of our galaxy, is about ten times the earth's orbital speed.

The idea of the ether had been so widely accepted that it was many years before it was finally abandoned. In fact two physicists, Fitzgerald and Lorentz, proposed to explain the null result of the Michelson-Morley experiment by suggesting that a body *contracts* in the direction of its motion through the ether in precisely the ratio $\sqrt{1 - u^2/c^2}$. This amount of shortening, known as the *Fitzgerald-Lorentz contraction,* would just equalize the two light paths so that there would be no fringe shift. Now such an ad hoc explanation of the experiment is not very satisfactory, for the contraction is not capable of direct observation. Any attempt to measure it would fail, since the measuring apparatus contracts along with the object to be measured.

I.2 Einstein's Postulates of Special Relativity

In 1905 Albert Einstein formulated his special theory of relativity. This theory is based on two fundamental postulates:

(1) *All physical laws have the same form in all inertial coordinate systems*
(2) *The speed of electromagnetic radiation in the vacuum is the same in all inertial systems*

The first postulate is a statement concerning physical laws in general and is an extension of Newtonian relativity. It can be shown that Maxwell's equations obey this postulate; that is, the equations have the same general form in any inertial coordinate system. The proof is given in almost any textbook on relativity [32].

The second postulate is more specific. It is the one that is of immediate application to our study of optics. It says that any measurement of the speed of light must always yield the same result, even if the source of light is in motion relative to the observer, or if the observer is moving relative to the source. This postulate immediately explains the null result of the Michelson-Morley experiment, for it implies that the speed of propagation of each beam in the experimental arrangement is always c, whether the apparatus is moving or not. Hence there is no phase change and no fringe shift.[1]

I.3 Relativistic Effects in Optics

According to the second postulate of the special theory of relativity, *the speed of light in vacuum is the same for any observer, regardless of the motion of the source relative to him or of his motion relative to the source.* To examine the consequences of this postulate, let us consider two observers moving with constant relative speed u. We shall designate the coordinate systems of our two observers by $Oxyz$ and $O'x'y'z'$, respectively. For simplicity, we shall assume that the respective axes Ox, $O'x'$, and so forth, are parallel, and that the relative motion is in the xx' direction (Figure I.3).

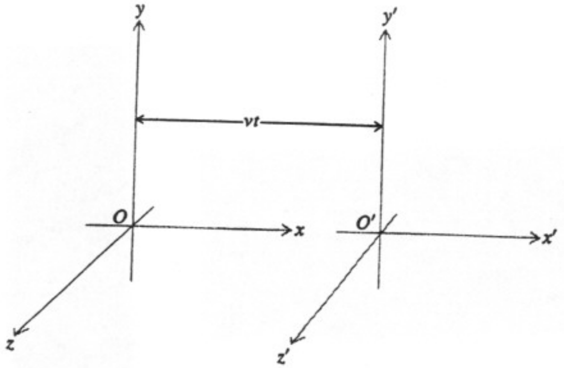

Figure I.3. Coordinate systems of two observers moving with constant relative speed.

Suppose that the two origins O and O' are coincident at time $t = 0$. Then the distance OO' is equal to ut, and the equations of transformation, according to classical or Newtonian kinematics, are

$$x = x' + ut'$$
$$y = y'$$
$$z = z'$$
$$t = t'$$

[1] It should be noted that Einstein did not formulate the theory of relativity in order to account for the Michelson-Morley experiment. Rather, the Michelson-Morley experiment has merely been cited as one experiment that tends to confirm the second postulate. Other, more recent experiments have been performed to verify the constancy of the velocity of light when the source and the observer are in relative motion. An excellent discussion and review of these is given by J. G. Fox, *Amer. J. Phys.*, **33**, 1 (1965).

The equation $t = t'$ expresses the assumed equality of the time scales of the two observers. They are using identical clocks. The above equations of transformation clearly contradict the second postulate, because from them we obtain $dx/dt = dx'/dt' + u$, so that anything moving with the speed of light c in, say, the primed system, moves with the speed $c + u$ in the unprimed system.

In order to find a coordinate transformation that agrees with the second postulate of relativity, let us consider the wave equation

$$\frac{\partial^2 U}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0$$

This is the differential equation of a light wave propagating with velocity c in the x direction. The requirement of the second postulate is that the equation remains invariant when referred to the primed coordinate system. That is,

$$\frac{\partial^2 U}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2 U}{\partial t'^2} = 0$$

This will be the case if

$$\frac{\partial^2 U}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2 U}{\partial t'^2} \tag{I.1}$$

Now it turns out that a general linear transformation[2] of the form

$$x = a_{11}x' + a_{12}t'$$
$$t = a_{21}x' + a_{22}t'$$

with the proper choice of constants will make the wave equation invariant. By substituting the above transformation in (I.1) and requiring that the equation be an identity, we obtain three equations to solve for the coefficients a_{11}, a_{12}, and so forth. We also need the subsidiary condition that $x = 0$ transforms to $x' = -ut'$, so that $a_{12} = ua_{11}$. The result is the famous Lorentz transformation,

$$\begin{aligned} x &= \gamma(x' + ut') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{ux'}{c^2}\right) \end{aligned} \tag{I.2}$$

where

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

[2] A nonlinear transformation would be unrealistic because uniform motion in one coordinate system would appear as accelerated motion in the other.

It is assumed that the kinematic consequences of the Lorentz transformation, for example, length contraction and time dilatation, are already familiar to the reader [32].

The Relativistic Doppler Formula Let us now consider a plane electromagnetic wave whose space–time dependence is of the form exp $i(kx - \omega t)$ when represented in the unprimed coordinate system. The observer in this system sees a wave of angular frequency $\omega = ck$ moving in the x direction. By applying the Lorentz transformation we find that the observer in the primed coordinate system sees the space–time dependence of this same wave as

$$\begin{aligned} &\exp i\left[k\gamma(x' + ut') - \omega\gamma\left(t' + \frac{ux'}{c^2}\right)\right] \\ &= \exp i\left[\left(k\gamma - \frac{\omega\gamma u}{c^2}\right) x' - (\omega\gamma - k\gamma u)\, t'\right] \end{aligned} \tag{I.3}$$

This must be identical with the expression

$$\exp i(k'x' - \omega' t')$$

Hence

$$\omega' = \omega\gamma\left(1 - \frac{ku}{\omega}\right) = \omega\gamma\left(1 - \frac{u}{c}\right) \tag{I.4}$$

Also, since $\omega = 2\pi\nu$ and $\gamma = [1 - (u^2/c^2)]^{-1/2}$, we can write

$$\nu' = \nu\,\frac{1 - u/c}{\sqrt{1 - u^2/c^2}} = \nu\,\frac{\sqrt{1 - u/c}}{\sqrt{1 + u/c}} = \nu\left(1 - \frac{u}{c} + \frac{1}{2}\frac{u^2}{c^2} - \cdots\right) \tag{I.5}$$

This is the relativistic Doppler formula. The series expansion shows that the relativistic Doppler shift differs from the nonrelativistic values only in the second- and higher-order terms, and therefore this difference becomes important only for large velocities. The relativistic formula has been verified by experiments with high-speed hydrogen atoms in a specially designed discharge tube shown in Figure I.4 [20].

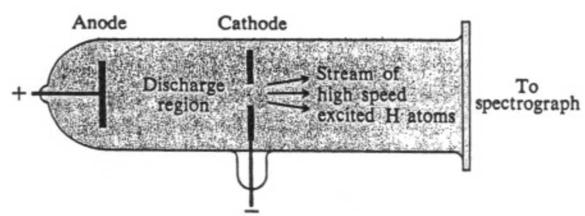

Figure I.4. Discharge tube used to observe the relativistic Doppler effect.

The Transverse Doppler Shift Let us next suppose that we have a plane wave traveling in the negative y direction in the unprimed system. The space–time dependence of this wave is exp $i(ky + \omega t)$. By applying the Lorentz transformation we find that to an observer in the primed system moving with speed v in the x direction, the space–time dependence of the wave is

$$\exp i\left[ky' + \omega\left(\gamma t' + \frac{ux'\gamma}{c^2}\right)\right] = \exp i\left[\frac{\omega u\gamma x'}{c^2} + ky' + \omega\gamma t'\right]$$

Since this must be the same as

$$\exp i\,(k_x'x' + k_y'u' + \omega' t')$$

then, for the coefficient of t' we have

$$\omega' = \omega\gamma$$

or, equivalently,

$$\nu = \nu'\sqrt{1 - \frac{u^2}{c^2}} = \nu'\left(1 - \frac{u^2}{2c^2} + \cdots\right) \tag{I.6}$$

This is the formula for the *transverse Doppler shift*, giving the frequency change when the relative motion is at right angles to the direction of observation. The transverse Doppler shift is a second-order effect and is therefore very difficult to measure. It has been verified by using the Mossbauer effect with gamma radiation from radioactive atoms [11].

The Aberration of Starlight Another consequence of the relativistic transformation of a plane wave moving in the y-direction is the appearance of x' in the wave function. This implies that the wave vector $\mathbf{k}'$ has a component in the x' direction and, consequently, the direction of propagation is not exactly the same as the direction of the y' axis. The angle α of the inclination to the y' axis is given by tan $\alpha = k_{x'}/k_{y'}$. Hence

$$\tan\alpha = \frac{\omega\gamma u/c^2}{k} = \frac{u}{c}\gamma = \frac{u}{c\sqrt{1 - u^2/c^2}} \tag{I.7}$$

This effect is known as *aberration of light*. It was first observed experimentally by the English astronomer Bradley in 1727. Bradley found an apparent shift in the positions of stars that was greatest for those stars whose line of sight was at right angles to the earth's orbital velocity around the sun. The maximum value of this stellar aberration is about 20 s of arc. Bradley's explanation is illustrated in Figure I.5, which shows the change in apparent direction due to the observer's velocity v. The situation is similar to that of a person running through

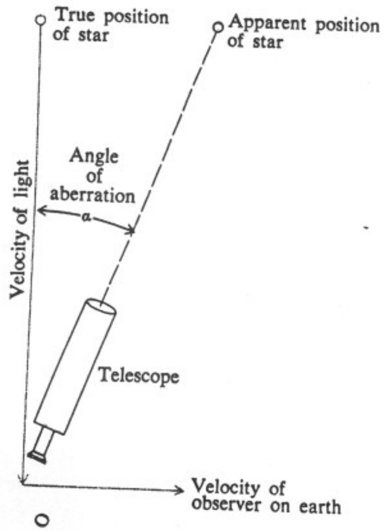

Figure I.5. The aberration of starlight.

falling rain. If the rain is falling straight down, its velocity relative to the person is not vertical but has a horizontal component equal to the forward speed of the person. From the figure we have tan $\alpha = u/c$. This simple formula differs from the relativistic formula (I.7) by the factor γ. In the case of the earth, however, u/c is of the order of 10^{-4}, so the difference is entirely negligible.

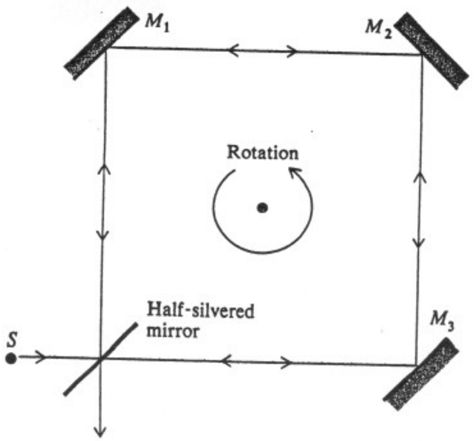

Figure I.6. Diagram of Sagnac's experiment.

It is interesting to note that the nonrelativistic transformation gives zero aberration for plane waves, hence aberration is a relativistic effect in this context. The simple explanation is valid if light is considered to be a hail of photons, however.

I.4 The Experiments of Sagnac and of Michelson and Gale to Detect Rotation

In 1911 the French physicist G. Sagnac performed an interesting experiment designed to detect rotation by means of light beams. His ex-

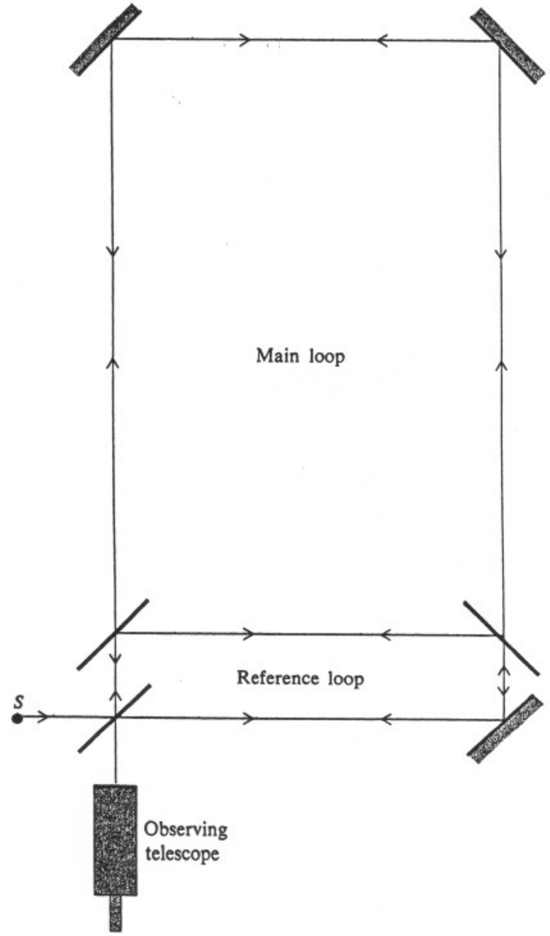

Figure I.7. The Michelson-Gale experiment for detecting the absolute rotation of the earth.

periment is illustrated in Figure I.6. A beam of light from a source S is divided into two beams by means of a half-silvered mirror M. The two beams are caused to traverse opposing paths around a circuit formed by mirrors M_1, M_2, and M_3 as shown. The beams recombine at M and are reflected into an observing telescope in which interference fringes are seen.

The apparatus is mounted on a rigid support that can be rotated about a vertical axis. The rotation causes a difference in the time required for the clockwise and counterclockwise beams to traverse the circuit. The result is a fringe shift that is proportional to the angular velocity of rotation. It is easy to show that the effective path difference Δs for the two beams is given approximately by

$$\Delta s = \frac{4A}{c}\,\Omega$$

where A is the area of the circuit and Ω is the angular velocity.

Sagnac was able to observe a fringe shift with a square light path about 1 m on a side and a speed of rotation of 120 r/min. In order to detect small angular velocities, a larger loop is required. In 1925 Michelson and Gale set up the experiment with a large path, 2/5 mile by 1/5 mile (Figure I.7). With this loop they were able to detect the expected fringe shift due to rotation of the earth. A smaller loop inside the larger one was used to provide a set of reference fringes.

REFERENCES

1. Beran, M. J., and G. B. Parrent, Jr., *Theory of Partial Coherence*. Englewood Cliffs, N. J.: Prentice-Hall, 1964.
2. Besancon, R. M., ed., *The Encyclopedia of Physics*. New York: Reinhold, 1966.
3. Bloembergen, N., *Nonlinear Optics*. New York: W. A. Benjamin, 1965.
4. Bloom, A. L., W. E. Bell, and F. O. Lopez, *Phys. Rev.*, **135**, A578 (1964).
5. Born, M., and E. Wolf, *Principles of Optics*. New York: Macmillan, 1964.
6. Boyd, G. D., and J. P. Gordon, *Bell System Tech. J.*, **40**, 489 (1961).
7. Candler, C., *Modern Interferometers*. London: Hilger and Watts, 1951.
8. Fowles, G. R., and W. T. Silfvast, *J. Quantum Electronics*, **QE-1**, 131 (1965).
9. Fox, A. G., and T. Li, *Bell System Tech. J.*, **40**, 453 (1961).
10. Francon, M., *Modern Applications of Physical Optics*. New York: Wiley, 1963.
11. Frauenfelder, H., *The Mössbauer Effect*. New York: W. A. Benjamin, 1962.
12. Gabor, D., *Nature*, **161**, 777 (1948).
13. Gordon, J. P., H. Z. Zeiger, and C. H. Townes, *Phys. Rev.*, **95**, 282 (1954).
14. Gray, D. E. ed., *American Institute of Physics Handbook*. New York: McGraw-Hill, 1957.
15. Hanbury-Brown, R., and R. Q. Twiss, *Proc. Roy. Soc.* (*London*), **A243**, 291 (1957).
16. Harnwell, G. P., *Principles of Electricity and Electromagnetism*. New York: McGraw-Hill, 1938.
17. Harrison, G. R., R. C. Lord, and J. R. Loofbourow, *Practical Spectroscopy*. Englewood Cliffs, N. J.: Prentice-Hall, 1948.
18. Herzberg, G., *Atomic Spectra and Atomic Structure*. New York: Dover, 1950.
19. ———, *Molecular Spectra and Molecular Structure*. Princeton, N. J.: Van Nostrand, 1950.
20. Ives, H. E., and G. R. Stilwell, *J. Opt. Soc. Am.*, **31**, 369 (1941).
21. Javan, A., W. R. Bennet, Jr., and D. R. Herriott, *Phys. Rev. Letters*, **6**, 106 (1961).
22. Jensen, R. C., and G. R. Fowles, *Proc. IEEE*, **52**, 1350 (1964).
23. King, G. W., *Spectroscopy and Molecular Structure*. New York: Holt, Rinehart and Winston, 1964.